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Polar Decomposition of Unit Dual Quaternions

This paper seeks to extend the notion of polar decomposition (PD) from matrix algebra to dual-quaternion algebra. The goal is to obtain a simple, efficient and explicit method for determining the PD of spatial displacements in Euclidean three-space that belong to a special Euclidean Group known as SE(3). It has been known that such a decomposition is equivalent to the projection of an element of SE(3) onto SO(4) that yields hyper spherical displacements that best approximate rigid-body displacements. It is shown in this paper that a dual quaternion representing an element of SE(3) can be decomposed into a pair of unit quaternions, called double quaternion, that represents an element of SO(4). Furthermore, this decomposition process may be interpreted as the projection of a point in four-dimensional space onto a unit hypersphere. An example is provided to illustrate that the results obtained from this dual-quaternion based polar decomposition are same as those obtained from the matrix based polar decomposition. [DOI: 10.1115/1.4024236]

1 Introduction

This paper deals with the problem of approximating a spatial displacement in Euclidean three-space E^3, i.e., an element of SE(3), with a rotation in Euclidean four-space E^4, i.e., an element of SO(4). While the idea that SE(3) is a limiting case of SO(4) has its roots in Physics [1,2], in kinematics it was first explored by McCarthy [3], wherein he used a combination of geometry and homogeneous transform to show how planar and spatial rigid-body motions can be regarded as special cases of spherical and 3-spherical motions, respectively. In recent years, this problem has been revisited in the context of searching for a more meaningful metric for spatial displacements for motion interpolation problem. Etzel and McCarthy [4,5] approximated spatial displacements in SE(3) with hyperspherical rotations in SO(4) to perform motion synthesis, and used Ge's [6] results on matrix realization of biquaternions [7] to define an approximately bi-invariant metric for spatial displacements. Tse and Larochelle [8] used the same approximation approach for spherical mechanism design. More recently, Larochelle et al. [9], Larochelle [10], and Larochelle and Murray [11] have used PD and singular value decomposition (SVD) of a homogeneous transformation matrix associated with an element of SE(n) to find the nearest approximating element in SO($n+1$) from SE(n). Belta and Kumar [12] used Lie Group to provide a firm mathematical basis to this problem and presented a SVD based approach for motion interpolation. Eberharter and Ravani [13] defined a metric for rigid-body displacements based on an optimized local mapping of the Study's quadric [14] via stereographic projections. In the context of PD and SVD based metric approaches, Shoemake and Duff's [15] work is of central importance. In this work, they proved that the orthogonal matrix resulting from PD is closest to the given matrix in the sense of Frobenius norm.

All the existing work in this area, however, have focused exclusively on the use of matrix based decomposition methods for motion approximation. Since the seminal work by Yang and Freudenstein [16,17] on the application of dual-quaternion algebra to the analysis of spatial mechanisms, dual-quaternion algebra has been widely recognized as an elegant tool for the treatment of kinematics of SE(3). In recent years, dual quaternions have found applicability in diverse areas. Radavelli et al. [18] have presented a comparison of dual-quaternion representation with Denavit-Hartenberg representation in kinematics of robotic system. Perez and McCarthy [19] have used dual quaternions for kinematic synthesis of robots. Kavan et al. [20] have used them in Computer Graphics and Animation for skinning of skeletally deformable models. In Aeronautics, Ma and Wang [21] have used dual quaternions for relative position and attitude estimation for close formation flights, while Yuanxin et al. [22] have used them for error characterization of strapdown inertial navigation system. In Control, Pham et al. [23] have performed position and orientation control of robot manipulators using dual-quaternion feedback, while Wang and Yu [24] and Han et al. [25] have given rise to control schemes for rigid-body transformation. In Physics, dual-quaternionic matrices have been formulated for solving Maxwell equations in electromagnetism [26].

Double quaternion algebra is an equally elegant tool for the treatment of double rotations of SO(4) [27]. Instead of matrix algebra, this paper seeks to develop a novel method for computing polar decomposition of SE(3) directly using dual quaternions and double quaternions. The result is a very simple and elegant method for polar decomposition that does not involve complicated numerical routines. Furthermore, this decomposition process may be interpreted geometrically as the projection of a point in four-dimensional space onto the unit hypersphere. This geometric interpretation does not involve dual-number projective space, which is the Image Space of dual quaternions introduced by Ravani and Roth [28].

The organization of the paper is as follows. Section 2 presents a review of the fundamentals of approximating spatial displacements by four-dimensional rotations as originally given by McCarthy [3]. Section 3 summarizes matrix based polar composition methods including SVD and PD. Section 4 presents the main contribution of this paper, which is the development of a dual-quaternion based approach to polar decomposition of SE(3). Section 5 provides a proof that the result obtained from the polar decomposition of dual quaternions is equivalent to the fact that SE(3) is a limiting case of SO(4). The proof involves the use of Plücker and Study coordinates for line transformations in SE(3) and SO(4). Section 6 provides a simple algorithm for polar decomposition based on dual quaternions and double quaternions. In the end, an example is provided to illustrate the effectiveness of this new algorithm.

2 SE(3) as a Limiting Case of SO(4)

This section reviews the work by McCarthy [3] in which he showed how spatial rigid-body motions can be regarded as special

Contributed by the Mechanisms and Robotics Committee of ASME for publication in the Journal of Mechanisms and Robotics. Manuscript received February 20, 2012; final manuscript received March 23, 2013; published online June 24, 2013. Assoc. Editor: Andrew P. Murray.

case of three-spherical motions. A spatial displacement that belongs to SE(3) can be given by a 4×4 homogeneous matrix which combines 3D rotation with translation. The following transformation represents a spatial displacement in the $W = R$ hyperplane

$$\begin{Bmatrix} X \\ Y \\ Z \\ W \end{Bmatrix} = [H] \begin{Bmatrix} x \\ y \\ z \\ R \end{Bmatrix} = \left[\begin{array}{ccc|c} & & & a/R \\ & [K(\theta,\phi,\zeta)] & & b/R \\ & & & c/R \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{Bmatrix} x \\ y \\ z \\ R \end{Bmatrix} \tag{1}$$

where (a, b, c) denotes the translation vector, and $[K(\theta, \phi, \zeta)]$ represents the 3×3 rotation matrix. On the other hand, a general rotation in Euclidean four-space E^4, which is an element of SO(4), is decomposed into a rotation (denoted by $K(\theta, \phi, \zeta)$) in the three-dimensional subspace (X–Y–Z), followed by three more rotations in W–X, W–Y, and W–Z planes with rotation angles α, β, γ, respectively. In this way, a general rotation in E^4, denoted by a transformation $[N]$, may be given by

$$\begin{Bmatrix} X \\ Y \\ Z \\ W \end{Bmatrix} = [N] \begin{Bmatrix} x \\ y \\ z \\ w \end{Bmatrix} = [J(\alpha,\beta,\gamma)] \left[\begin{array}{ccc|c} & & & 0 \\ & [K(\theta,\phi,\zeta)] & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{Bmatrix} x \\ y \\ z \\ w \end{Bmatrix} \tag{2}$$

where

$$[J(\alpha,\beta,\gamma)] = \begin{bmatrix} c\alpha & 0 & 0 & s\alpha \\ -s\beta s\alpha & c\beta & 0 & s\beta c\alpha \\ -s\gamma c\beta s\alpha & -s\gamma s\beta c\gamma & c\gamma & s\gamma c\beta c\alpha \\ -c\gamma c\beta s\alpha & -s\beta c\gamma & -s\gamma & c\gamma c\beta c\alpha \end{bmatrix} \tag{3}$$

In the above equation, c and s represent the cosine and sine functions, respectively. Now consider a small 4D rotation near the point $(0, 0, 0, R)$ where R is now interpreted as the radius of a hypersphere. The rotation angles in W–X, W–Y, and W–Z planes may be given by

$$\tan\alpha = \frac{a}{R}, \quad \tan\beta = \frac{b}{R}, \quad \tan\gamma = \frac{c}{R} \tag{4}$$

Taylor series expansion of Eq. (2) after substitution of Eq. (4) leads to

$$\begin{Bmatrix} X \\ Y \\ Z \\ R \end{Bmatrix} = [H] \begin{Bmatrix} x \\ y \\ z \\ R \end{Bmatrix} + \frac{1}{R} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline e_1 & e_2 & e_3 & 0 \end{array} \right] \begin{Bmatrix} x \\ y \\ z \\ R \end{Bmatrix} + O\left(\frac{1}{R^2}\right) \tag{5}$$

where $[H]$ denotes the 4×4 homogeneous transform associated with a spatial displacement in E^3 as given in Eq. (1).

In Eq. (5), we have $e_i = -(k_{1i}a + k_{2i}b + k_{3i}c)$ with $i = 1, \ldots, 3$ and k_{ij} $(i, j = 1, 2, 3)$ denote the elements of matrix $[K(\theta, \phi, \zeta)]$. The second term in Eq. (5) has nonzero elements only in the direction perpendicular to the $W = R$ hyperplane. Thus, displacements in E^3 can be regarded as limiting cases for rotations in E^4 as R increases.

Inversely, with a spatial displacement given, one may obtain the corresponding 4D rotation matrix following Etzel and McCarthy's [4] simple algorithm:

(1) Select an R from $R = L/(\xi)^{1/2}$, where L can be chosen to be the largest of the components of translation vectors, and ξ is a user-defined measure of error in approximation;
(2) Obtain values of angles α, β, γ from Eq. (4);
(3) Obtain $[N] \in SO(4)$ by determining $[J(\alpha, \beta, \gamma)]$ and $[K(\theta, \phi, \psi)]$ for each displacement (Eq. (2)).

3 Polar Decomposition of Homogeneous Transform

Belta and Kumar [12] studied the approximation problem between SE(3) and SO(4) as a projection from SE(3) to SO(4). The 4D rotation matrix as an element of SO(4) was obtained by applying the SVD method to the homogeneous transform $[H]$. Recently, Larochelle et al. [9] applied the PD method to the matrix $[H]$.

The SVD of $[H]$ yields three factors, $[H] = [U][\Sigma][V]^T$, where $[U]$ and $[V]$ are orthonormal matrices and $[\Sigma]$ is diagonal matrix with positive elements. The PD results in $[H] = [P][S]$, where $[S]$ is a positive definite symmetric matrix and $[P]$ is another orthogonal matrix. These orthogonal matrices are related by $[P] = [U][V]^T$. It has been shown that $[P]$ is the closest possible orthogonal matrix to $[H]$ in the sense of the Frobenius norm; see for example, Shoemake and Duff [15]. However, it is well-known that both PD and SVD are generally expensive to compute with no explicit expressions available to calculate them.

It is noted here that the translation terms in the homogeneous transform $[H]$ are divided by R, which is the radius of the approximating hypersphere. This makes $[H]$ dimensionless, which is desirable from the viewpoint of motion approximation. While mathematically one may apply PD to a homogenous transform without the factor R, the resulting rotation matrix $[P]$ will be dependent on the choice of the unit for the translation component. For example, if the unit "inch" is used instead of "feet," the numerical value of the translation vector (a, b, c) will be scaled by a factor of 12 and this in turn changes the value of the matrix $[H]$ obtained from $[H] = [P][S]$. Angeles [29] discussed this issue in the context of "characteristic length L" for motion approximation. Larochelle and McCarthy [30,31] suggested the use of $L = 24 l_{\max}/\pi$ (where $l_{\max}$ is the largest translation component) as a normalizing factor for the translation vector (a, b, c).

4 Polar Decomposition of Unit Dual Quaternions

This section follows the seminal work by Yang and Freudenstein [16,17] as well as classical theory on elliptic geometry [27] to develop a novel method for computing polar decomposition of SE(3) directly using dual quaternions and double quaternions.

First, let us take a look at the well-known analogy between the polar decomposition of a matrix, $[H] = [P][S]$, and the polar form of a nonzero complex number, $z = re^{i\theta}$. Both cases involve rotation operators ($e^{i\theta}$ for 2D rotation and $[P]$ for 4D rotation) as well as stretching factors (positive scalar r for complex number and positive symmetric $[S]$ for matrix). Furthermore, a general quaternion, $\mathbf{T} = r\mathbf{Q}$, is also decomposed into a scaling factor r and a unit quaternion

$$\mathbf{Q} = e^{\mathbf{s}(\theta/2)} = \cos(\theta/2) + \mathbf{s}\sin(\theta/2) \tag{6}$$

that corresponds to a 3D rotation[1] about the axis $\mathbf{s} = s_1 i + s_2 j + s_3 k$ where $(1, i, j, k)$ form the quaternion basis. Likewise,

[1]More precisely, a unit quaternion $\mathbf{Q}$ corresponds to a special equal-angle double rotation in 4D. It is the composition of $\mathbf{Q}$ and its conjugate $\mathbf{Q}^*$ that results in a 3D rotation in a subspace of 4D with rotation angle θ.

a general dual quaternion, $\hat{\mathbf{T}} = \hat{r}\hat{\mathbf{Q}}$, consists of a dual-number scaling factor $\hat{r} = r + \varepsilon r^0$ and a unit dual quaternion $\hat{\mathbf{Q}} = \mathbf{Q} + \varepsilon\mathbf{Q}^0$, where the symbol ε denotes a dual-number unit with the property $\varepsilon^2 = 0$. The real part, $\mathbf{Q}$, is a unit quaternion representing a 3D rotation and the dual part is given by the following quaternion product:

$$\mathbf{Q}^0 = (1/2)\mathbf{d}\mathbf{Q} = (d/2)\bar{\mathbf{d}}\mathbf{Q} \tag{7}$$

where $\mathbf{d} = d\bar{\mathbf{d}}$ is a vector quaternion representing the vector of translation with d being the magnitude of the vector and $\bar{\mathbf{d}} = \bar{d}_1 i + \bar{d}_2 j + \bar{d}_3 k$ being a unit vector quaternion, i.e., $\bar{d}_1^2 + \bar{d}_2^2 + \bar{d}_3^2 = 1$. It is easy to verify that the real and dual parts satisfy the condition

$$Q_1 Q_1^0 + Q_2 Q_2^0 + Q_3 Q_3^0 + Q_4 Q_4^0 = 0 \tag{8}$$

where (Q_1, Q_2, Q_3, Q_4) are components of quaternion $\mathbf{Q}$ and $(Q_1^0, Q_2^0, Q_3^0, Q_4^0)$ are components of $\mathbf{Q}^0$. Details on dual-quaternion algebra can be found in Ref. [32].

We now present a simple method for converting a unit dual quaternion into a pair of unit quaternions (**G**, **H**) called double quaternion. Following McCarthy [33], we identify dual unit ε to be equal to $1/R$ due to the property that $\varepsilon^2 = 1/R^2 = 0$. The effect of making this substitution in the following amounts to dividing the translation components by R as in Eq. (1). This leads to

$$\mathbf{Q} + \varepsilon\mathbf{Q}^0 = \mathbf{Q} + \frac{1}{2R}\mathbf{d}\mathbf{Q} = \left(1 + \frac{d}{2R}\bar{\mathbf{d}}\right)\mathbf{Q} = r\mathbf{D}\mathbf{Q} \tag{9}$$

where the scaling factor r is given by

$$r = \frac{\sqrt{4R^2 + d^2}}{2R} \tag{10}$$

and $\mathbf{D}$ is a unit quaternion given by

$$\mathbf{D} = \frac{2R}{\sqrt{4R^2 + d^2}} + \bar{\mathbf{d}}\frac{d}{\sqrt{4R^2 + d^2}} \tag{11}$$

Similarly, we have

$$\mathbf{Q} - \varepsilon\mathbf{Q}^0 = \mathbf{Q} - \frac{1}{2R}\mathbf{d}\mathbf{Q} = \left(1 - \frac{d}{2R}\bar{\mathbf{d}}\right)\mathbf{Q} = r\mathbf{D}^*\mathbf{Q} \tag{12}$$

where r is given by Eq. (10) and $\mathbf{D}^*$ is the conjugate of the unit quaternion $\mathbf{D}$

$$\mathbf{D}^* = \frac{2R}{\sqrt{4R^2 + d^2}} - \bar{\mathbf{d}}\frac{d}{\sqrt{4R^2 + d^2}} \tag{13}$$

Introducing a pair of unit quaternions, **G** and **H**, such that

$$\mathbf{G} = \mathbf{D}\mathbf{Q}, \quad \mathbf{H} = \mathbf{D}^*\mathbf{Q} \tag{14}$$

It follows from Eqs. (9) and (12) that

$$\mathbf{Q} + \varepsilon\mathbf{Q}^0 = r\mathbf{G}, \quad \mathbf{Q} - \varepsilon\mathbf{Q}^0 = r\mathbf{H} \tag{15}$$

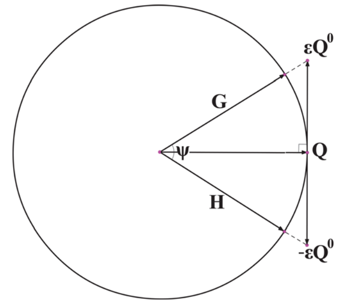

Fig. 1 A double quaternion (G, H) that approximates a dual quaternion ($\mathbf{Q}, \varepsilon\mathbf{Q}^0$) is obtained by projecting points $\mathbf{Q} + \varepsilon\mathbf{Q}^0$ and $\mathbf{Q} - \varepsilon\mathbf{Q}^0$ onto the unit hypersphere

where r is the scaling factor given by Eq. (10). We call Eq. (15) as the *polar decomposition* of a dual quaternion $\mathbf{Q} + \varepsilon\mathbf{Q}^0$. It is well known in elliptic geometry [27] that a pair of unit quaternions (**G**, **H**) defines a double rotation in E^4 and is therefore called a double quaternion.

Introducing an angle ψ such that

$$\cos(\psi/2) = \frac{2R}{\sqrt{4R^2 + d^2}}, \quad \sin(\psi/2) = \frac{d}{\sqrt{4R^2 + d^2}} \tag{16}$$

geometrically, we can consider Eq. (15) as the projection of the points defined by $\mathbf{Q} \pm \varepsilon\mathbf{Q}^0$ onto the unit hypersphere (Fig. 1). Thus, as R increases, r in the limit converges to 1 and the dual quaternions $\mathbf{Q} \pm \varepsilon\mathbf{Q}^0$ converge to double quaternion (**G**, **H**). Since a dual quaternion represents an element of SE(3) and a double quaternion represents an element of SO(4), this result is consistent with the fact that SE(3) is a limiting case of SO(4) as discussed in Sec. 2.

To establish an equivalence with the matrix based PD method, we need to obtain the matrix form of the 4D rotation. Let $\mathbf{G} = G_4 + G_1 i + G_2 j, G_3 k$ and $\mathbf{H} = H_4 + H_1 i + H_2 j + H_3 k$, then the rotation matrix $[N]$ resulting from the polar decomposition $[H] = [N][S]$ is given by $[N] = [G^+][H^*]^-$ where (see Ref. [6])

$$[G^+] = \begin{bmatrix} G_4 & -G_3 & G_2 & G_1 \\ G_3 & G_4 & -G_1 & G_2 \\ -G_2 & G_1 & G_4 & G_3 \\ -G_1 & -G_2 & -G_3 & G_4 \end{bmatrix} \tag{17}$$

$$[H^*]^- = \begin{bmatrix} H_4 & -H_3 & H_2 & -H_1 \\ H_3 & H_4 & -H_1 & -H_2 \\ -H_2 & H_1 & H_4 & -H_3 \\ H_1 & H_2 & H_3 & H_4 \end{bmatrix} \tag{18}$$

5 Line Transformations in SE(3) and SO(4)

In this section, we use the polar decomposition of dual quaternions (Eq. (15)) to show that line transformation in SE(3) is a limiting case of line transformation in SO(4). This provides an independent proof that the polar decomposition of dual quaternions (15) is equivalent to the fact that SE(3) is a limiting case of SO(4).

It is well known that the geometry of 4D rotations is equivalent to the geometry of elliptic three-space and that a line in elliptic three-space (which is equivalent to a great circle in E^4) can be represented either by Plücker coordinates or Study coordinates [27]. Let $\hat{\mathbf{p}} = \mathbf{p} + \varepsilon\mathbf{p}^0$ denote the dual vector in Plücker form. Their corresponding normalized study vectors that represent the same line are related to the Plücker form via an equation similar to Eq. (15)

$$\mathbf{p} + \varepsilon \mathbf{p}^0 = w\mathbf{u}, \quad \mathbf{p} - \varepsilon \mathbf{p}^0 = w\mathbf{v} \tag{19}$$

where $\varepsilon = 1/R$ and w is the scaling factor given by

$$w = \frac{\sqrt{R^2 + \mathbf{p}^0 \cdot \mathbf{p}^0}}{R} \tag{20}$$

Equation (19) is here referred to as *polar decomposition of line coordinates* that convert from Plücker vectors to Study vectors. Similar to the relationship between dual and double quaternion illustrated geometrically in Fig. 1, the geometric relationship between the Plücker vectors and Study vectors is such that Study vectors can be regarded as projection of Plücker vectors onto the unit sphere. Now consider a transformation of line coordinates in elliptic three-space. Let ($\mathbf{G}$, $\mathbf{H}$) denote a pair of unit quaternions that represent a general displacement in the elliptic three space (which is equivalent to a double rotation in E^4). In terms of Study vectors, the line coordinate transformation in elliptic three space is given by the following quaternion equations [27]:

$$\tilde{\mathbf{u}} = \mathbf{G}\mathbf{u}\mathbf{G}^{-1}, \quad \tilde{\mathbf{v}} = \mathbf{H}\mathbf{v}\mathbf{H}^{-1} \tag{21}$$

where $\mathbf{G}^{-1} = \mathbf{G}^*/|\mathbf{G}|$ is the inverse of $\mathbf{G}$. For a unit quaternion, we have $|\mathbf{G}| = 1$ and therefore, $\mathbf{G}^{-1}$ is equal to its conjugate $\mathbf{G}^*$. As the above transformations preserve the norm of the vectors, it follows that

$$w\tilde{\mathbf{u}} = \mathbf{G}(w\mathbf{u})\mathbf{G}^{-1} \tag{22}$$

Substituting Eqs. (19) into (22), we obtain

$$\tilde{\mathbf{p}} + \varepsilon \tilde{\mathbf{p}}^0 = \mathbf{G}(\mathbf{p} + \varepsilon \mathbf{p}^0)\mathbf{G}^{-1} \tag{23}$$

This is equivalent to

$$\tilde{\mathbf{p}} + \varepsilon \tilde{\mathbf{p}}^0 = (r\mathbf{G})(\mathbf{p} + \varepsilon \mathbf{p}^0)(r\mathbf{G})^{-1} \tag{24}$$

where r is a nonzero scaling factor. Replacing $r\mathbf{G}$ with $\mathbf{Q} + \varepsilon \mathbf{Q}^0$ in Eq. (24), we obtain, after some algebra

$$\tilde{\mathbf{p}} + \varepsilon \tilde{\mathbf{p}}^0 = (\mathbf{Q} + \varepsilon \mathbf{Q}^0)(\mathbf{p} + \varepsilon \mathbf{p}^0)(\mathbf{Q}^* + \varepsilon (\mathbf{Q}^0)^*) \tag{25}$$

Expanding the right side of Eq. (25) and letting $\varepsilon^2 = 0$, we obtain, for the real part

$$\tilde{\mathbf{p}} = \mathbf{Q}\mathbf{p}\mathbf{Q}^* \tag{26}$$

and for the dual part

$$\tilde{\mathbf{p}}^0 = \mathbf{Q}\mathbf{p}(\mathbf{Q}^0)^* + \mathbf{Q}\mathbf{p}^0\mathbf{Q}^* + (\mathbf{Q}^0)\mathbf{p}\mathbf{Q}^* \tag{27}$$

Equations (26) and (27) constitute the well-known rule for dual-quaternion algebra [34] that define the transformation of line coordinates in SE(3). Following the same derivation using $\tilde{\mathbf{v}} = \mathbf{H}\mathbf{v}\mathbf{H}^{-1}$, the second equation in Eq. (21), will yield the same rule. This completes the proof that dual and double quaternions in the polar decomposition (15) represent a spatial displacement $[H]$ of SE(3) and a double rotation $[N]$ of SO(4), respectively, and that the resulting spatial displacement $[H]$ is a limiting case of $[N]$.

6 The Algorithm and Examples

Here is a summary of our dual-quaternion based PD method for obtaining the rotation matrix $[N]$:

(1) For a given spatial displacement $[H]$, first convert the 3×3 rotation matrix to its quaternion representation $\mathbf{Q}$ and use the translation vector $\mathbf{d} = (d_1, d_2, d_3)$ to obtain $\mathbf{Q}^0$ using the quaternion product $\mathbf{Q}^0 = (1/2)\mathbf{d}\mathbf{Q}$.

(2) Select a sufficiently large R to define $\varepsilon = 1/R$. Then, use polar decomposition of dual quaternions as given by Eq. (15) to obtain $\mathbf{G}$ and $\mathbf{H}$.

(3) Use Eqs. (17) and (18) to compute the rotation matrix $[N] = [G^+][H^*]^-$.

From a computational perspective, it is important to answer the question of *largeness* of the hypersphere and to establish a relationship, if any, between the radius R of the hypersphere and the choice of the normalizing length used in matrix based PD approach of Larochelle et al. [9]. The intent is to show that matrix polar decomposition of SE(3), an otherwise nonlinear and computationally expensive process, can be performed completely analytically using double quaternions.

We have seen earlier in Sec. 3 that the polar factor $[N]$ results from a matrix polar decomposition of homogeneous matrix as $[H] = [N][S]$. Using the orthogonality of $[N]$ and the symmetry of $[S]$, we can write

$$\begin{aligned} [H] &= [N][S], \quad \text{or} \\ [N^T][H] &= [S], \quad \text{and} \\ [N^T][H] &= [H^T][N] \end{aligned} \tag{28}$$

On the other hand, we have seen before that the approximation errors in representing an element of SE(3) by SO(4) are entirely due to the translation components (Eq. (1)). Therefore, we can examine the effects of translation by constructing a homogeneous matrix in SE(3) parameterized by translation components only. Without any loss of generality, consider a homogeneous matrix $[H]$ containing a single translation component normalized by the characteristic length L as in Larochelle et al. [30,31]

$$[H] = \begin{bmatrix} 1 & 0 & 0 & a/L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

The above matrix can be approximated as a 4D rotation in the W–X hyperplane, which can be written as

$$[N] = [G^+][H^*]^- = [J(\alpha, 0, 0)] \left[\begin{array}{ccc|c} & [K(0,0,0)] & & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \tag{30}$$

where $\tan \alpha = a/R$, $[K] = [I]$, and

$$[J(\alpha, 0, 0)] = \begin{bmatrix} c\alpha & 0 & 0 & s\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\alpha & 0 & 0 & c\alpha \end{bmatrix} \tag{31}$$

Substituting for $[N]$ and $[H]$ from above in the last equation of Eq. (28) and solving, we get $\tan \alpha = a/(2L)$. Since, $\tan \alpha = a/R$, we obtain

$$R = 2L \tag{32}$$

The same relationship is obtained for other translation components. This is a deceptively simple result that firmly places matrix based PD and dual-quaternion based PD on the same ground. What this means is that one can now simply choose a sufficiently large hypersphere radius R equal to twice the normalizing length proposed by Larochelle and colleagues, and then use our

Table 1 **Four spatial locations from Larochelle et al. [9]**

No.	(a, b, c)	(θ, ϕ, ζ) (all angles in degree)
1	(0.00, 0.00, 0.00)	(0.0, 0.0, 0.0)
2	(0.00, 1.00, 0.25)	(15.0, 15.0, 0.0)
3	(1.00, 2.00, 0.50)	(45.0, 60.0, 0.0)
4	(2.00, 3.00, 1.00)	(45.0, 80.0, 0.0)

Table 2 **Frobenius Norm Error**

Displacement no.	$\|[H] - [G^+][H^*]^-\|_F$
1	0
2	0.03180
3	0.07071
4	0.11553

algorithm to get the same hyperspherical rotation matrix ($[G^+][H^*]^-$) as produced by matrix based polar decomposition. The issue of choice of a characteristic length L was settled by Angeles [29], and thus due to the relationship in Eq. (32), it is settled for the choice of hypersphere's radius as well. He showed that there is no optimum choice of a characteristic length, yet a length can be *engineered*. Thus, the errors of approximation decrease monotonically with increasing length or hypersphere radius.

Now, we present a numerical example. Table 1 gives spatial displacements from an example in Larochelle et al. [9]. Following are the 4D rotation matrix $[N_i]$ obtained using our dual-quaternion based PD algorithm.

$$[N_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[N_2] = \begin{bmatrix} 0.9659 & -0.0670 & 0.2500 & 0 \\ 0.0000 & 0.9657 & 0.2587 & 0.0218 \\ -0.2588 & -0.2501 & 0.9330 & 0.0055 \\ 0.0014 & -0.0197 & -0.0107 & 0.9997 \end{bmatrix},$$

$$[N_3] = \begin{bmatrix} 0.7070 & -0.6124 & 0.3530 & 0.0218 \\ -0.0000 & 0.5000 & 0.8649 & 0.0436 \\ -0.7071 & -0.6124 & 0.3533 & 0.0109 \\ -0.0077 & -0.0018 & -0.0493 & 0.9988 \end{bmatrix},$$

$$[N_4] = \begin{bmatrix} 0.7068 & -0.6956 & 0.1212 & 0.0436 \\ -0.0005 & 0.1748 & 0.9824 & 0.0654 \\ -0.7073 & -0.6960 & 0.1220 & 0.0218 \\ -0.0154 & 0.0342 & -0.0724 & 0.9967 \end{bmatrix}$$

The results are identical to using a built-in MATLAB function for implementation of matrix based PD. We use Larochelle et al.'s [30,31] formula for characteristic length $L = 24 l_{\max}/\pi$, where $l_{\max}$ is the largest translation component among all the given displacements. In this example, the length calculates to $L = 22.9183$ and the hypersphere radius is $R = 2L = 45.8366$.

Table 2 gives the Frobenius norm of error between respective elements of SE(3) and SO(4) for the current example. Once again, the results were in agreement with matrix based PD computation up to at least seven digits of precision.

7 Conclusions

In this paper, we have extended the notion of matrix algebra based polar decomposition to a dual-quaternion based approach. The merits of the approach are in having a simple and analytical formulation as opposed to resorting to numerical, computationally expensive, black-box approach for calculation of PDs. We also showed an intuitive geometric interpretation of connection between dual and double quaternions and exploited this connection to establish an efficient algorithm for PD calculations. The results have critical applications in ascertaining a meaningful measure of distance between rigid-body displacements for motion approximation and interpolation problems.

Acknowledgment

This work was supported by National Science Foundation under Grant No. CMMI-0856594.

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