On the exact computation of the swept surface of a cylindrical surface undergoing two-parameter rational Bézier motions

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Abstract: In this paper, we present exact analysis and representation of the swept surfaces of a cylindrical object undergoing two-parameter rational Bézier motions. Instead of using the well-known approach for analysing the point trajectory of an object motion for swept volume analysis, this paper seeks to develop a new method for swept volume analysis by studying the plane trajectory of a two-parameter rational motion. It brings together recent work in swept volume analysis, plane representation of developable surfaces, and computer aided synthesis of two-parameter freeform rational motions. The results have applications in design and approximation of freeform surfaces as well as in tool path planning for five-axis machining of freeform surfaces.

Keywords: rational motions; swept volume; enveloping surfaces; cylindrical cutter; multi-axis NC machining.

Reference to this paper should be made as follows: Xia, J., Ge, Q.J. and Purwar, A. (2011) 'On the exact computation of the swept surface of a cylindrical surface undergoing two-parameter rational Bézier motions', *Int. J. Mechatronics and Manufacturing Systems*, Vol. 4, Nos. 3/4, pp.356–369.

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1 Introduction

For the past two decades, swept volume analysis has received a great deal of attention due to its applications in manufacturing automation such as tool motion generation, verification and simulation for NC machines and robot motion planning with obstacle avoidance. For NC applications, the problem of swept volume analysis has been studied almost exclusively in the context of one-parameter motion (see, for example, Chappel, 1983; Bobrow, 1985; Ganter, 1985; Wang and Wang, 1986a, 1986b; Sungurtekin and Voelcker, 1986; Jerard and Drysdale, 1988; Choi and Jun, 1989; Weld and Leu, 1990; Marciniak, 1991; Blackmore and Leu, 1992a, 1992b; Menon and Voelcker, 1993; Li and Jerard, 1994; Hu and Ling, 1994; Sarma and Dutta, 1997; Lee, 1997; Pottmann and Wallner, 1999). Abdel-Malek et al. (2006) have presented an excellent summary on the representation and analysis of swept volumes. In a related development, Radzevich (2007) has presented kinematic geometry of optimal surface machining using multi-axis NC machines. Most of the methods for swept volume calculations are approximate in nature, and even the exact methods have been attempted only for one-parameter motions.

This paper deals with the problem of analysing the swept volume of a cylinder undergoing a two-parameter motion. The notion of analytically defined two-parameter motion was discussed in Bottema and Roth (1979). Ge and Sirchia (1999) developed computer aided design methods for synthesising two-parameter freeform motions such as rational Bézier and B-spline motions, and Purwar et al. (2006) provided methods for automatic fairing of such motions.

Ge et al. (1998) studied the plane trajectory of a two-parameter rational Bézier motion. Xia and Ge (2001) presented an exact representation of the swept surfaces of a cylindrical cutter under rational Bézier and B-spline motions by considering the cylindrical surface as the envelope of a one-parameter family of planes. Based on that work and Xia (2001), recently Bi et al. (2010) presented analytical envelope surface representation of a conical cutter undergoing one-parameter rational motion. Zhang et al. (2005) studied generation of tool path for five-axis free-form surface machining using rational Bézier motions of a flat-end cutter.

Other related efforts in this area are due to Jüttler and Wagner (1999), Pottmann et al. (1999), Pottman and Walner (2000), Eberharter and Ravani (2004) and Sprott and Ravani (2002, 2008), who brought together geometry and kinematics for the shape and surface design.

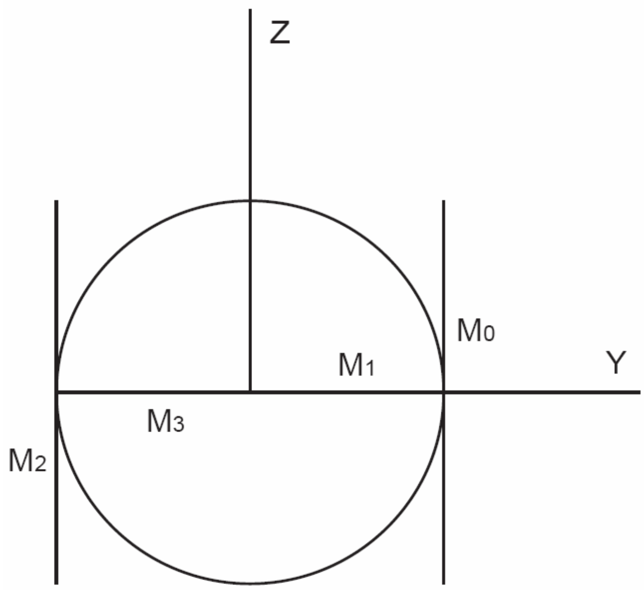
The present paper brings together these recent threads of research and develops a method for computing the swept surface of a cylindrical surface under a two-parameter rational Bézier motion. The resulting surfaces are referred to as kinematic surfaces, and they can be used for kinematics driven shape design and approximation. The goal of combining geometry and kinematics in shape design is to bridge the gap between design and manufacturing of geometric shapes in computer aided environment. Much of the existing work in computer aided geometric design (CAGD) is based on point geometry and does not take into account the kinematics of rigid body motion. While the resulting curve and surface representations are highly suitable for computer rendering and interactive design of freeform (or sculptured) surfaces in a CAD environment, physical shapes of freeform surfaces are manufactured with geometric features of a physical object (the cutter) under the coordinated motion of a machine tool such as a computer numerical controlled (CN) machine. For example, dies, molds, and other sculpture-surfaced parts commonly used in automobile and aerospace industries are typically produced with three- to five-axis CNC machining processes, which are usually followed by manual or robotic grinding and polishing. In order to produce a sculptured surface with such manufacturing processes, it is necessary to convert the static and point-geometric based CAD model of the surface into a kinematic model of the surface, i.e., in terms of the swept surface of the cutter motion. The use of such surfaces in sculptured shape design has the potential to greatly reduce the difficulty in handling the kinematic and geometric issues involved in tool/workpiece interaction in manufacturing automation such as NC machining and robot motion planning.

The organisation of the paper is as follows. Section 2 briefly reviews how a circular cylinder may be represented as a two-piece rational quadratic developable surfaces using Bézier control planes as opposed to control points. Section 3 presents the trajectory of a plane undergoing a rational Bézier motion of degree ($2n$, $2m$). Section 4 introduces the boundary surfaces of the swept volume of a cylindrical tool undergoing a rational Bézier motion of degree ($2n$, $2m$). Section 5 deals with the problem of computing the enveloping surfaces of a cylindrical surface undergoing a rational Bézier motion of degree ($2n$, $2m$). We, then present an example which computes various types of surfaces generated by different parts of a cylindrical tool.

2 Rational representation of a cylinder using plane coordinates

This section presents a rational representation of a circular cylinder as a two-piece rational quadratic developable surface. Each of the developable surfaces is represented using its plane coordinates as opposed to point coordinates. The resulting representation is known as dual representation of the developable surface. It has been shown by Xia and Ge (2001) that the representation combined with rational motions reduces the problem of computing a point on the swept surface of the cylindrical surface into that of computing the intersection of three planes. This leads to computationally efficient methods for analysing a swept surface without the need to invoke the enveloping condition that requires the surface normal to be perpendicular to the velocity at a point.

Figure 1 Four control planes for the cylindrical surface viewed along the X-axis



For a cylindrical tool with radius r whose axis is along the X-axis (Figure 1), the homogeneous coordinates of three control planes that can represent the upper half (above Y-axis) of the cylindrical surface are given by

$$\mathbf{M}_0 = (0,1,0,-r), \mathbf{M}_1 = (0,0,1,0), \mathbf{M}_2 = (0,-1,0,-r). \tag{1}$$

The family of the tangent planes of the half cylinder above Y-axis can be expressed by the following rational Bézier representation:

$$\mathbf{M}^2(s) = \sum_{i=0}^{2} B_i^2(s)\mathbf{M}_i. \tag{2}$$

where $B_i^2(s)$ is Bernstein polynomial.

For $s \in [0, 1]$, equation (2) defines the top half of the cylinder. For $s < 0$ and $s > 1$, equation (2) defines the bottom half of the cylinder. In order to avoid potential numerical problems when s becomes very large, the bottom half of the cylinder is instead represented by another set of three control planes. The first and the last control planes are the same as before, and the middle control plane coincides with $\mathbf{M}_1$ but has opposite orientation, i.e., $\mathbf{M}_3 = -\mathbf{M}_1 = (0, 0, -1, 0)$.

3 Two-parameter rational Bézier Motions

This section summarises how a two-parameter rational motion in Bézier form can be represented using dual quaternions. Details can be found in Ge and Sirchia (1999) and Purwar et al. (2006). The notion of using quaternions and dual quaternions to study displacements and motions has been a subject of study in many papers and a number of textbooks such as Bottema and Roth (1979) and McCarthy (1990). The key work on the free-form rational motion design can be found in Ge and Ravani (1994a, 1994b) and in Jüttler and Wagner (1996), while a comprehensive survey on using quaternions and dual quaternions to represent rigid body motions can be found in Röschel (1998) and Purwar and Ge (2005).

The basic idea is to represent a spatial displacement using a set of eight homogeneous coordinates known as the dual quaternion components or the dual Euler parameters. In this way, a spatial displacement may be considered as a point in the space of dual quaternions. A one-parameter dual quaternion curve in the space corresponds to a one-parameter motion. A two-parameter surface in the space of dual quaternions corresponds to a two-parameter motion. Freeform curves and surfaces in the space of dual quaternions such as Bézier curves and tensor-product Bézier surfaces correspond to one-and two-parameter freeform motions, respectively, in rational Bézier form in the Cartesian space.

For a given dual quaternion $\hat{\mathbf{Q}} = (\mathbf{Q}, \mathbf{R})$, the corresponding rigidity preserving transformations of point and plane coordinates are given by (see Ge and Sirchia, 1999; Purwar and Ge, 2005 for a derivation):

$$\tilde{\mathbf{P}} = \mathbf{QPQ}^* + \left(\mathbf{RQ}^* - \mathbf{QR}^*\right) \tag{3}$$

$$\tilde{\mathbf{M}} = \mathbf{QMQ}^* + \mathbf{RmQ}^* - \mathbf{QmR}^* \tag{4}$$

where '*' denotes the conjugate of a quaternion, **P** and **M** represent homogeneous point and plane coordinates, respectively, in the quaternion form, and **m** is the vector part of the quaternion $\mathbf{M} = (\mathbf{m}, m_4)$.

Given a set of dual quaternions $\hat{\mathbf{Q}}_i = (\mathbf{Q}_i, \mathbf{R}_i)$, where $0 \le i \le n$, the following pair of equations

$$\begin{aligned} \mathbf{Q}(t) &= \sum\nolimits_{i=0}^{n} B_i^n(t)\mathbf{Q}_i, \\ \mathbf{R}(t) &= \sum\nolimits_{i=0}^{n} B_i^n(t)\mathbf{R}_i, \end{aligned} \tag{5}$$

define a Bézier curve of degree n in the space of dual quaternions. Substituting equation (5) into (3) and (4), we obtain the point trajectories of the motions as rational Bézier curves of degree $2n$, while the plane trajectories of the motion are obtained as rational developable surfaces of degree $2n$. Therefore, The Bézier dual-quaternion curve of degree n corresponds to a rational Bézier motion of degree $2n$ in the Cartesian space.

Similarly, given an $n \times m$ array of dual quaternions $\hat{\mathbf{Q}}_{i,j} = (\mathbf{Q}_{i,j}, \mathbf{R}_{i,j})$, the following pair of equations

$$\begin{aligned} \mathbf{Q}^{n,m}(u,v) &= \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{Q}_{i,j} B_i^n(u) B_j^m(v) \\ \mathbf{R}^{n,m}(u,v) &= \sum_{k=0}^{n}\sum_{l=0}^{m} \mathbf{R}_{k,l} B_k^n(u) B_l^m(v), \end{aligned} \tag{6}$$

define a tensor-product Bézier surface of degree (n, m) in the space of dual quaternions, which corresponds to a two-parameter rational Bézier motion of degree ($2n$, $2m$). In this paper, we are interested in obtaining an explicit analytical matrix-form representation of the trajectory of a moving plane **M** undergoing such a two-parameter motion.

The trajectory of a moving plane **M** is obtained by substituting expressions for $\mathbf{Q}^{n,m}(u, v)$ and $\mathbf{R}^{n,m}(u, v)$ from equation (6) into equation (4) (see Ge et al., 1998 for details):

$$\tilde{\mathbf{M}}^{2n,2m}(u,v)=\left[H^{2n,2m,*}(u,v)\right]\mathbf{M}.$$

where the 4 × 4 matrix $[H^{2n,2m,*}(u, v)]$ represents the transformations of the plane **M** under the two-parameter motion (6). The bi-variate matrix function can be put in tensor-product Bézier form as

$$\left[H^{2n,2m,*}(u,v)\right]=\sum_{g=0}^{2n}\sum_{h=0}^{2m}B_g^{2n}(u)B_h^{2m}(v)\left[H_{g,h}^{*}\right], \tag{7}$$

where $\left[H_g^{*},h\right]$ denote the Bézier control matrices and they are given by

$$\left[H_{g,h}^{*}\right]=\sum_{i+k=g}\sum_{j+l=h}\frac{C_i^nC_j^mC_k^nC_l^m}{C_g^{2n}C_h^{2m}}\left[H_{i,j,k,l}^{*}\right].$$

Here, C_i^n denote the binomial coefficients and

$$\left[H_{i,j,k,l}^{*}\right]=\left[Q_{ij}^{+}\right]\left[Q_{kl}^{-}\right]+\left[Q_{kl}^{-}\right]\left[R_{ij}^{m+}\right]-\left[Q_{ij}^{+}\right]\left[R_{kl}^{m-}\right], \tag{8}$$

with

$$\left[Q_{ij}^{+}\right]=\begin{bmatrix} Q_{ij,4} & -Q_{ij,3} & Q_{ij,2} & Q_{ij,1} \\ Q_{ij,3} & Q_{ij,4} & -Q_{ij,1} & Q_{ij,2} \\ -Q_{ij,2} & Q_{ij,1} & Q_{ij,4} & Q_{ij,3} \\ -Q_{ij,1} & -Q_{ij,2} & -Q_{ij,3} & Q_{ij,4} \end{bmatrix}, \tag{9}$$

$$\left[Q_{kl}^{-}\right]=\begin{bmatrix} Q_{kl,4} & -Q_{kl,3} & Q_{kl,2} & -Q_{kl,1} \\ Q_{kl,3} & Q_{kl,4} & -Q_{kl,1} & -Q_{kl,2} \\ -Q_{kl,2} & Q_{kl,1} & Q_{kl,4} & -Q_{kl,3} \\ Q_{kl,1} & Q_{kl,2} & Q_{kl,3} & Q_{kl,4} \end{bmatrix}, \tag{10}$$

$$\left[R_{ij}^{m+}\right]=\begin{bmatrix} R_{ij,4} & -R_{ij,3} & R_{ij,2} & 0 \\ R_{ij,3} & R_{ij,4} & -R_{ij,1} & 0 \\ -R_{ij,2} & R_{ij,1} & R_{ij,4} & 0 \\ -R_{ij,1} & -R_{ij,2} & -R_{ij,3} & 0 \end{bmatrix}, \text{ and} \tag{11}$$

$$\left[R_{kl}^{m-}\right]=\begin{bmatrix} R_{kl,4} & -R_{kl,3} & R_{kl,2} & 0 \\ R_{kl,3} & R_{kl,4} & -R_{kl,1} & 0 \\ -R_{kl,2} & R_{kl,1} & R_{kl,4} & 0 \\ R_{kl,1} & R_{kl,2} & R_{kl,3} & 0 \end{bmatrix}. \tag{12}$$

In the above, $Q_{ij,k}$ and $R_{ij,k}$ (k = 1,2,3,4) denote the components of quaternions $\mathbf{Q}_{i,j}$ and $\mathbf{R}_{i,j}$, respectively.

The plane trajectory represents a two-parameter continuous family of tangent planes which envelops a tensor-product surface, referred to as the swept surface of a plane or a *kinematic dual tensor-product surface*. An alternative way to evaluate a tangent plane with parameter (u, v) on the swept surface is to use the two-parameter de Casteljau algorithm. Let $\left[H_{0,0}^{2n-1,2m-1,*}\right]\mathbf{M}$, $\left[H_{0,1}^{2n-1,2m-1,*}\right]\mathbf{M}$, $\left[H_{1,0}^{2n-1,2m-1,*}\right]\mathbf{M}$ and $\left[H_{1,1}^{2n-1,2m-1,*}\right]\mathbf{M}$ denote four de Casteljau planes for the last step of the de Casteljau algorithm. These four planes intersect at the same point, which is a point on the swept surface. Thus, any three of the four planes can be used to compute the intersection point.

4 Boundary surface classification

For a circular cylinder undergoing a two-parameter rational Bézier motion (Figure 2 shows the control quaternions of a Bézier motion of degree 3 × 3), the boundary surfaces of the swept volume of the motion fall into the following three categories:

Type I Boundary surfaces defined by the four cylinders at (u, v) = (0, 0), (0, 1), (1, 0) and (1, 1).

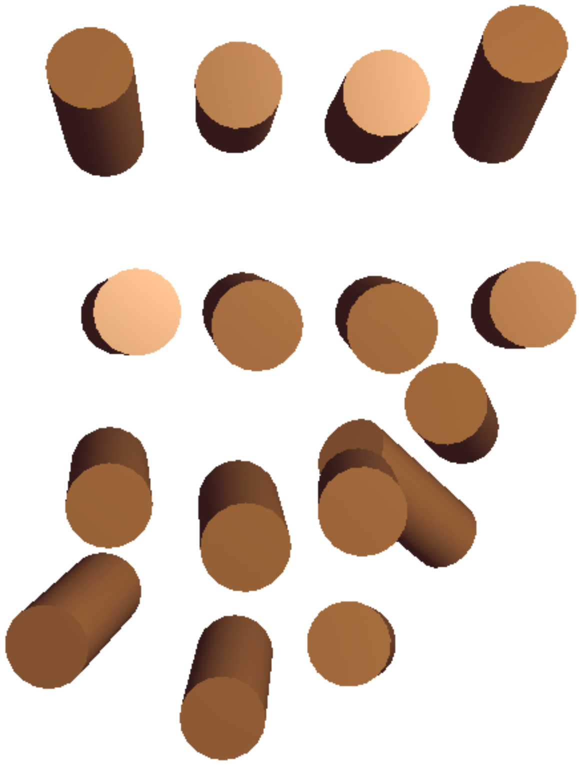
Type II Boundary surfaces defined by one-parameter boundary motions $\hat{\mathbf{Q}}(0,v)$, $\hat{\mathbf{Q}}(1,v)$, $\hat{\mathbf{Q}}(u,0)$ and $\hat{\mathbf{Q}}(u,1)$. These boundary surfaces can be obtained exactly using methods described in Xia and Ge (2001).

Type III

- IIIa The swept surface of the cylindrical surface undergoing the two-parameter Bézier motion.
- IIIb The swept surfaces of top and bottom circular planes undergoing the two-parameter Bézier motion.
- IIIc The swept surfaces of circular edges undergoing the two-parameter Bézier motion.

Boundary surfaces of the type III(b) are the kinematic dual where its Bézier dual quaternions are tensor product surfaces which have been studied in Ge et al. (1998). Boundary surfaces of type III(c) are simply two-parameter sweeps of circles, which can be obtained using the boundary surface of the type (IIIa).

Figure 2 Control quaternions of a rational Bézier motion of degree 3 × 3 (see online version for colours)



5 The swept surface of a cylindrical surface undergoing Two-parameter Rational Bézier Motions

Xia and Ge (2001) have studied the problem of computing the swept surface of a cylindrical surface under a one-parameter rational Bézier motion. Since the cylindrical surface is modelled by a one-parameter family of planes, the swept surface is represented directly as the enveloping surface of a two-parameter family of planes where one parameter s is associated with the cylindrical surface, and the other parameter t is associated with the rational motion. The resulting surface is a kinematic rational tensor-product surface in plane coordinates. With the aid of the two-parameter de Casteljau algorithm (see Ge and Sirchia, 1999), the characteristic point on the swept surface can be evaluated as the intersection of any of the four intermediate planes at the last step of the de Caseljau algorithm. For any given instant t, these characteristic points form a curve on the cylindrical surface called the characteristic curve of the sweep. This plane based approach allows the evaluation of the characteristic curve without invoking the classical enveloping condition $\mathbf{n} \cdot \mathbf{v} = 0$ where $\mathbf{n}$ is the surface normal at a given point and $\mathbf{v}$ is the velocity vector at the same point.

In this section, we develop an exact method for computing the swept surface of a cylindrical surface $\mathbf{M}(s)$ undergoing a two-parameter rational Bézier motion $\mathbf{Q}^{m,n}(u, v)$.

The swept surface of the two-parameter motion can be considered as the enveloping surface of a three-parameter (s, u, v) family of planes. For a two parameter motion, if we fix v and let u vary, we have a one-parameter motion, then a characteristic curve $\mathbf{P}^u(s, u, v)$ can be generated from this one-parameter motion. Similarly, a characteristic curve $\mathbf{P}^v(s, u, v)$ can be obtained by fixing u and varying v. Since normal to the surface at a point on $\mathbf{P}^u(s, u, v)$ is perpendicular to the u-direction velocity vector $\mathbf{V}_u$, and normal at a point on $\mathbf{P}^v(s, u, v)$ is perpendicular to the v-direction velocity vector $\mathbf{V}_v$, where $\mathbf{V}(\mathbf{V}_u, \mathbf{V}_v)$ is the velocity vector, the normal at the intersection point of the above two curves must be perpendicular to $\mathbf{V}$. Then, for a given set of parameter values (u_0, v_0), the corresponding position of the cylinder is the intersection of the following two *iso-parametric* motions in u-and v-directions, respectively:

$$\hat{\mathbf{Q}}^m(u) = \sum_{i=0}^{m} B_i^m(u)\hat{\mathbf{Q}}_i$$

where its Bézier dual quaternions are

$$\hat{\mathbf{Q}}_i = \sum_{j=0}^{n} B_j(v)\hat{\mathbf{Q}}_{i,j}$$

and

$$\hat{\mathbf{Q}}^n(v) = \sum_{j=0}^{n} B_j^n(v)\hat{\mathbf{Q}}_j$$

where its Bézier dual quaternions are

$$\hat{\mathbf{Q}}_j = \sum_{i=0}^{m} B_i(u)\hat{\mathbf{Q}}_{i,j}.$$

Since each of the above one-parameter motions yields a rational characteristic curve on the cylinder, the intersection of these two rational curves gives the characteristic point of the two-parameter motion at the given parameter values (u, v). In what follows, we summarise the steps of the algorithm for computing the characteristic point.

Let us first consider the one-parameter motion in the u-direction, i.e., we fix the variable v and let u vary. Let $\mathbf{M}_0^1(s)$ and $\mathbf{M}_1^1(s)$ denote two intermediate planes obtained by applying the one-parameter de Casteljau algorithm for the cylindrical surface $\mathbf{M}^2(s)$. Then from Xia and Ge (2001), we conclude that the four planes

$$\begin{aligned} &\left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0^1(s), \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1^1(s), \\ &\left[H_{0,0}^{2n-1,2m,*}(u,v)\right]\mathbf{M}^2(s), \left[H_{1,0}^{2n-1,2m,*}(u,v)\right]\mathbf{M}^2(s) \end{aligned} \tag{13}$$

are concurrent and that they intersect at a common point $\mathbf{P}^u(s, u, v)$. Taking the wedge product of the first three of the four planes, and after simple algebraic manipulations, we obtain

$$\begin{aligned}\mathbf{P}^u(s,u,v) &= (1-s)^4\mathbf{m}_0^u(u,v) + 2s(1-s)^3\mathbf{m}_1^u(u,v)\\ &\quad + (1-s)^2 s^2\mathbf{m}_2^u(u,v) + s(1-s)^3\mathbf{m}_3^u(u,v)\\ &\quad + 2s^2(1-s)^2\mathbf{m}_4^u(u,v) + s^3(1-s)\mathbf{m}_5^u(u,v)\\ &\quad + s^2(1-s)\mathbf{m}_6^u(u,v) + 2s^3(1-s)^2\mathbf{m}_7^u(u,v)\\ &\quad + s^4\mathbf{m}_8^u.\end{aligned} \tag{14}$$

where $\mathbf{m}_i^u(u,v)(i=0,...,8)$ depend only on the motion parameters (u, v) and are given by

$$\begin{aligned}\mathbf{m}_0^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1\\ &\quad \wedge \left[H_0^{2n,2m-1,*}(u,v)\right]\mathbf{M}_0,\\
\mathbf{m}_1^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_1,\\
\mathbf{m}_2^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_2,\\
\mathbf{m}_3^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_2\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_0,\\
\mathbf{m}_4^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_2\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_1,\\
\mathbf{m}_5^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_2\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_2,\\
\mathbf{m}_6^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_2\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_0,\\
\mathbf{m}_7^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_2\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_1,\\
\mathbf{m}_8^u &= \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1 \wedge \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_2\\ &\quad \wedge \left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}_2.\end{aligned}$$

The above equation can be rearranged to obtain

$$\mathbf{P}^u(s,u,v) = \sum_{i=0}^{4}\mathbf{f}_i^u(u,v)s^i, \tag{15}$$

where

$$\mathbf{f}_4^u = \mathbf{m}_0^u - \left(2\mathbf{m}_1^u + \mathbf{m}_3^u\right) + \left(\mathbf{m}_2^u + 2\mathbf{m}_4^u + \mathbf{m}_6^u\right) - \left(\mathbf{m}_5^u + 2\mathbf{m}_7^u\right) + \mathbf{m}_8^u$$

$$\mathbf{f}_3^u = -4\mathbf{m}_0^u + 3\left(2\mathbf{m}_1^u + \mathbf{m}_3^u\right) - 2\left(\mathbf{m}_2^u + 2\mathbf{m}_4^u + \mathbf{m}_6^u\right) + \left(\mathbf{m}_5^u + 2\mathbf{m}_7^u\right)$$

$$\mathbf{f}_2^u = 6\mathbf{m}_0^u - 3\left(2\mathbf{m}_1^u + \mathbf{m}_3^u\right) + \left(\mathbf{m}_2^u + 2\mathbf{m}_4^u + \mathbf{m}_6^u\right)$$

$$\mathbf{f}_1^u = -4\mathbf{m}_0^u + \left(2\mathbf{m}_1^u + \mathbf{m}_3^u\right)$$

$$\mathbf{f}_0^u = \mathbf{m}_0^u$$

As s varies in [0, 1], equation (15) represents a characteristic curve of one-parameter motion in the u-direction.

Similarly, for a one-parameter motion in the v-direction, we can conclude that the four Planes

$$\begin{aligned}&\left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0^1(s), \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1^1(s),\\&\left[H_0^{2n,2m-1,*}(u,v)\right]\mathbf{M}^2(s), \left[H_{1,0}^{2n,2m-1,*}(u,v)\right]\mathbf{M}^2(s)\end{aligned} \tag{16}$$

are concurrent and that they intersect at a point $\mathbf{P}^v(s, u, v)$. Note that the first two planes of the group given in equation (16) are the same as the first two planes of the group given in equation (13). As s varies from 0 to 1, the point traces out a curve, which is the characteristic curve of the one-parameter motion in v-direction. The equation for this characteristic curve is obtained using a procedure similar to the above as:

$$\mathbf{P}^v(s,u,v) = \sum_{i=0}^{4} \mathbf{f}_i^v(u,v)s^i, \tag{17}$$

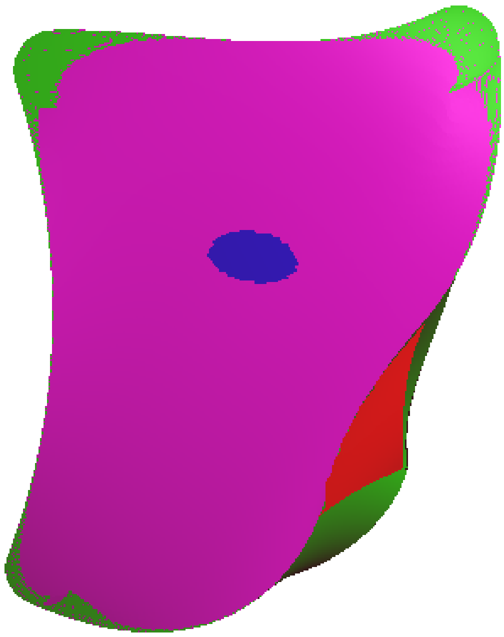
where $\mathbf{f}_i^v$ take the same form as $\mathbf{f}_i^u(u,v)$.

The intersection of the two characteristic curves, $\mathbf{P}^u(s, u, v)$ and $\mathbf{P}^v(s, u, v)$ yields the characteristic point of the cylindrical surface at the parameter values (u, v) for the two-parameter motion. The point of intersection can be obtained by relating the geometric parameter s to the motion parameter u and v. This can be achieved by requiring that the two groups of the planes given in equation (13) and equation (16), be concurrent at the same point. This is equivalent to requiring that any three of the four planes from one group is concurrent with any one of the two non-overlapping planes from the other group. This condition for concurrency can be expressed such that the determinant of the 4 × 4 matrix formed by the four homogeneous vectors representing the four selected planes vanishes. Here, we choose to require the determinant of the following four vectors to vanish:

$$\begin{aligned}&\left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_0^1(s), \left[H^{2n,2m,*}(u,v)\right]\mathbf{M}_1^1(s),\\&\left[H_0^{2n-1,2m,*}(u,v)\right]\mathbf{M}^2(s), \left[H_{1,0}^{2n,2m-1,*}(u,v)\right]\mathbf{M}^2(s).\end{aligned} \tag{18}$$

This results in a polynomial equation of degree 6 in the variable s. Using a symbolic computation software, we are able to show that there exists a common factor $(s^2 - s + 1)$ in the equation. Eliminating this common factor, we obtain a quartic equation in s. A method on how to find an exact solution to a quartic equation can be found in the Abramowitz and Stegun (1972).

Figure 3 Swept surfaces generated by a cylindrical surface under a two-parameter rational Bézier motion of degree (3, 3) (see online version for colours)



Notes: Type II (green) surface is generated by the four one-parameter boundary motions, Type IIIa (red) surface is generated by the cylindrical surface, while IIIb (blue) and IIIc (magenta) are generated by the top and bottom circular planes, and the circular edges, respectively.

After having solved for the s for one half of the cylindrical surface, we do not need to recompute for the other half, because both halves have the same value of s at the same instant (u, v). Figure 3 shows an example of the swept surfaces generated by a cylindrical surface undergoing a two-parameter rational Bézier motion of degree (3, 3).

6 Conclusions

In this paper we presented a method for the exact analysis of the swept surface of a cylindrical surface undergoing a two-parameter rational Bézier motion. The method extends the exact analysis and representation of the swept surfaces of a cylinder under a one-parameter rational Bézier motion. These methods provide a basis for the development of a kinematics-based methodology for the integration of CAD and CAM.

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