

Interactive Dimensional Synthesis and Motion Design of Planar 6R Single-Loop Closed Chains via Constraint Manifold Modification

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In this paper, we present an interactive, visual design approach for the dimensional synthesis of planar 6R single-loop closed chains for a given rational motion using constraint manifold modification. This approach is implemented in an interactive software tool that provides mechanism designers with an intuitive way to determine the parameters of planar mechanisms, and in the process equips them with an understanding of the design process. The theoretical foundation of this work is based on representing planar displacements with planar quaternions, which can be seen as points in a special higher dimensional projective space (called the image space), and on formulating the kinematic constraints of closed chains as algebraic surfaces in the image space. Kinematic constraints under consideration limit the positions and orientation of the coupler in its workspace. In this way, a given motion of a mechanism in the Cartesian space maps to a curve in the image space that is limited to stay within the bounds of the algebraic surfaces. Thus, the problem of dimensional synthesis is reduced to determine the parameters of equations that describe algebraic surfaces. We show that the interactive approach presented here is general in nature, and can be easily used for the dimensional synthesis of any mechanism for which kinematic constraints can be expressed algebraically. The process of designing is fast, intuitive, and especially useful when a numerical optimization based approach would be computationally demanding and mathematically difficult to formulate. This simple approach also provides a basis for students and early designers to learn and understand the process of mechanism design by simple geometric manipulations. [DOI: 10.1115/1.4001775]

1 Introduction

Typical task-driven problems, also known as motion synthesis problems, seek to synthesize the mechanisms that guide a rigid body to navigate through, or as close as possible to a prescribed motion, which is usually specified as either a set of finite or a continuous time variant sequence of displacements. The problem that we solve here is as follows: Given a prescribed planar rational motion, what are the dimensions of the links of a planar 6R closed chain that accomplishes the given task?

Theory of mechanism synthesis is well-developed (see Refs. [1–3]), and there have been a lot of academic research efforts in the development of software systems for the synthesis of planar, spherical, and spatial mechanisms (see KINSYN III from Rubel and Kaufman [4], LINCAGES from Erdman and co-workers [5,6] and Kihonge et al. [7], SPADES from Larochelle [8], SPHINX from Larochelle et al. [9], SPHINXPC from Ruth and McCarthy [10], OSIRIS from Tse and Larochelle [11], Perez and McCarthy [12], and Su and McCarthy [13], and SYNTHETICA from Su et al. [14]). In the commercial domain, SYMECH [15] and WATT [16] are two well-known software systems for planar mechanism design. Despite the availability of design packages, their implementation and interface for design of rigid body linkages mostly utilize a black-box-like approach, which involves inputting the desired motion, hiding the design theories and algorithms, and outputting a

mechanism, which makes the design procedures less intuitive. In this work, we have attempted to give rise to an intuitive design environment wherein the designer is not only able to synthesize a mechanism interactively by simple geometric manipulations in a higher dimensional space, but also learns to understand the relationship between the modifiable geometry and its apparent effect on the dimensions of the links.

In this paper, we use planar quaternions to represent displacements; see Refs. [17,18] for details on quaternion representation. Using the well-known kinematic mapping approach [19,20], the given rational motion is transformed into a rational curve in the space of quaternions (also known as the image space of displacements). For details on rational motions, see Refs. [21–27].

The general approach of the work presented in this paper is closely related to the kinematic mapping approach for dimensional synthesis of planar and spherical mechanisms pioneered by Ravani and Roth [19,20]. However, the kinematic mapping was originally introduced by Blaschke [28] and Grünwald [29] independently. Ravani and Roth's approach involved minimizing the distance error between the given displacements and the image curve of the chain. Their work was followed by Bodduluri and McCarthy [30], Bodduluri [31], and Larochelle [32]. Burmester [33] showed that a four-bar linkage can interpolate at most five given displacements exactly. For more than five displacements, usually approximation is required. This resulted in approximate motion synthesis. Hayes et al. [34] used kinematic mapping for preliminary development of an algorithm for the approximate synthesis of planar four-bar mechanisms for rigid body guidance. Brunnthaler et al. [35] used kinematic mapping to solve the prob-

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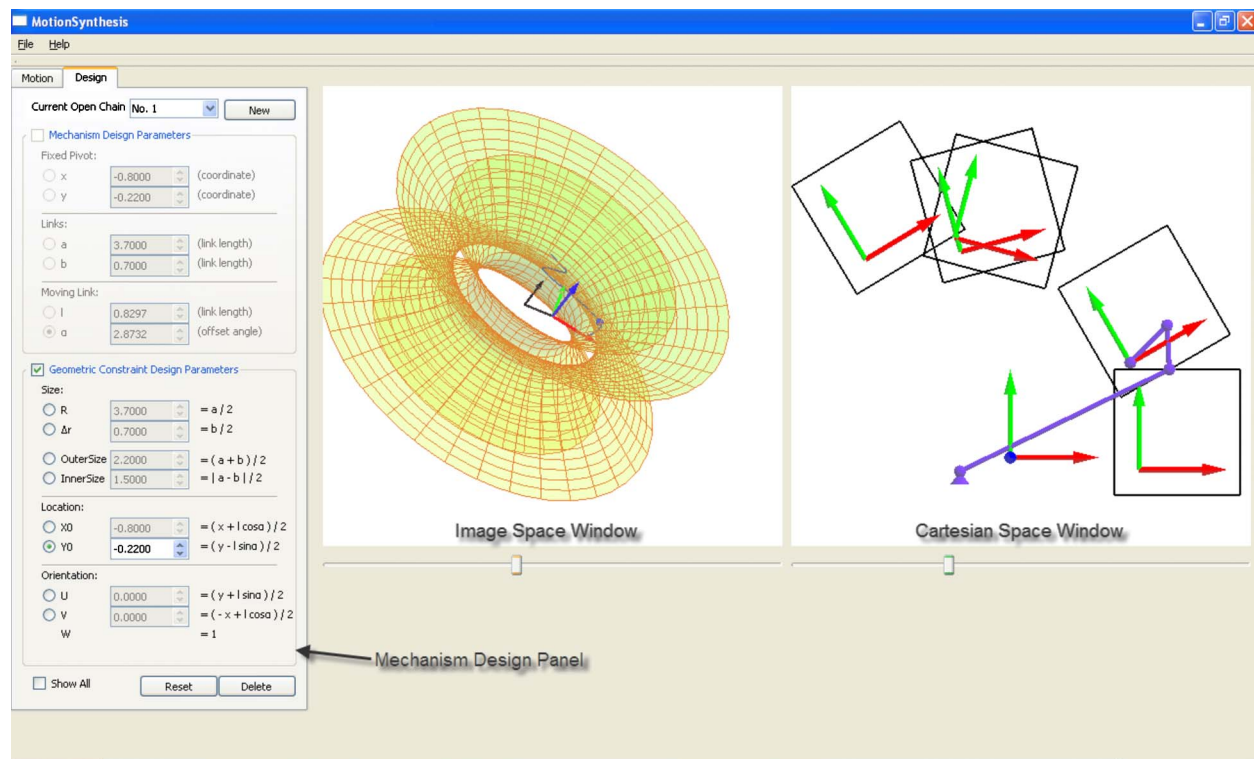


Fig. 1 A screenshot of the panels and the window space

lem of designing a spherical four-bar mechanism that interpolates a coupler through five given orientations. Venkataramanujam and Larochelle [36] used a parametrized constraint manifold and employed nonlinear optimization to give numerical methods for approximate motion synthesis of spherical open and closed chains. Here, we are employing the kinematic mapping for designing planar $6R$ closed chains that do not have a restriction on the number of interpolating displacements. This is feasible since the planar $6R$ closed chains have three degrees of freedom, and thus admit an infinite number of solutions.

This paper seeks to study the dimensional synthesis problem from the perspective of constrained motion interpolation. Jin and Ge [37,38] and Purwar et al. [39–41] studied the problem of motion interpolation under kinematic constraints for planar and spherical $2R$, $3R$ open, and $6R$ closed chains as well as spatial $5S$ chains. By using quaternions and kinematic mapping approach they transformed the problem of constrained motion interpolation into designing a rational curve constrained to fit the constraint manifold of the chain. Starting with an initial unconstrained curve, they modify the curve using an iterative numerical method until it fits inside the constraint manifold. The current paper investigates the inverse problem, that is, to change the constraint manifold while keeping the given rational curve fixed for the dimensional synthesis of planar $6R$ closed chains.

Our design method treats the $6R$ closed chains as mechanisms assembled using two open chains connected together at the ends. Each open chain imposes kinematic constraints that limit the positions and orientations of the object connected to the end link. In this paper, we use the algebraic form of the constraint manifold [18,42] for the planar $6R$ closed chains. Thus, the kinematic constraints are transformed into geometric constraints, and the given rational motion is transformed into a rational curve in the image space. This way, our problem reduces to finding the constraint manifold that accommodates the given rational curve. Algebraically, the kinematic constraints are derived in the inequality form, where the limits of the inequalities are functions of link lengths, while the constraint functions themselves incorporate parameters

that describe the location of fixed and moving pivots and the location and orientation of the moving frame. However, instead of dealing with the mechanism parameters directly, we formulate our approach in terms of the geometric parameters of the constraint manifold. This, and the decoupled nature of the relationship between various geometric parameters and the mechanism parameters, lends intuitiveness to our approach. We provide a design tool with a user-friendly interface (see Figs. 1 and 2), using which the designer can interactively change parameters that govern the size, orientation, and the position of the constraint manifold in the 4D image space (we visualize the 4D space by projecting on a hyperplane) such that the manifold contains the input image curve. This tool also provides a way to change the image curve in case the design requirements can be relaxed. In the end, we design two open chains that simultaneously satisfy the kinematic constraints and the motion requirements. A visual interpretation of this approach is that we try to find a close fit constraint manifold that will contain the given image curve entirely. The process of designing is fast, intuitive, and especially useful when a numerical optimization based approach would be computationally demanding and mathematically difficult to formulate. We do not attempt to find an optimal solution, but a good solution that satisfies the input motion requirements. Our simple approach also provides a basis for students and early designers to learn and understand the designing of mechanisms by simple geometric manipulations.

The organization of the paper is as follows. Section 2 reviews the concept of quaternions for planar displacements and the rational B -spline motions. Section 3 reviews the kinematic constraints of planar $6R$ closed chains using quaternions. Section 4 presents a method for the interactive dimensional synthesis of planar $6R$ closed chains, and establishes relationships between mechanism parameters and the manifold size, orientation, and position. Finally, we present an example and visualizations that demonstrate the method before giving concluding remarks.

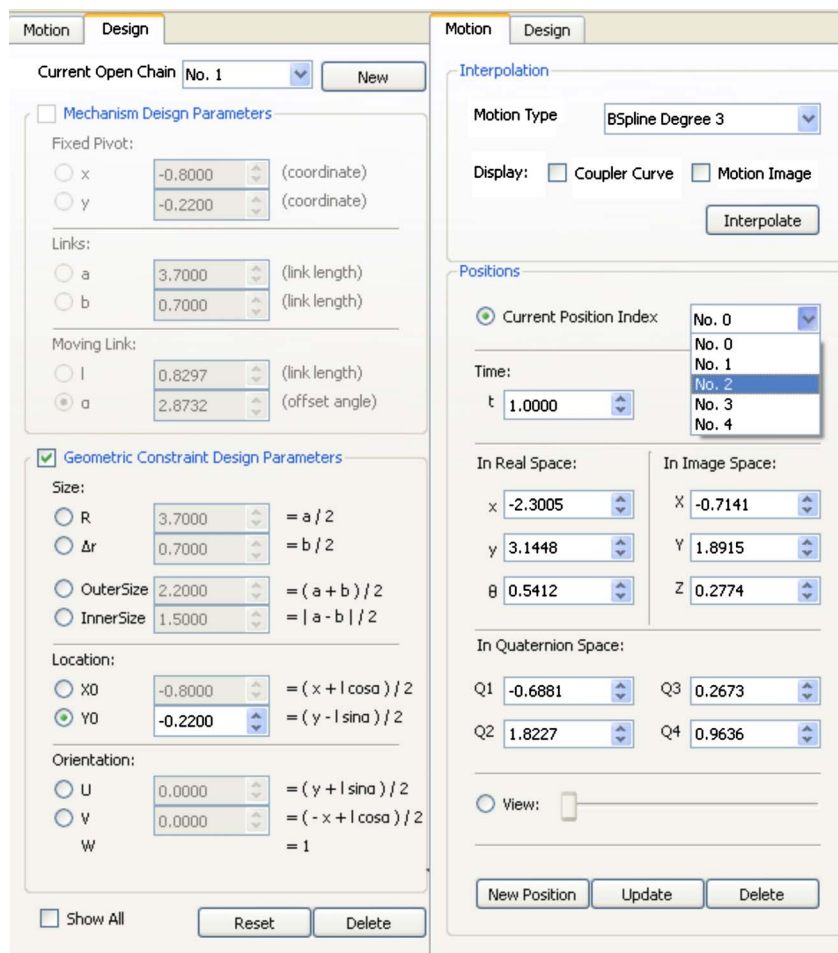


Fig. 2 A screenshot of the mechanism and motion design panels

2 Planar Quaternion and Rational Motion

In this section, we review the concept of planar quaternions and the free-form rational motion insofar as necessary for the development of the current paper.

2.1 Planar Quaternion. A planar displacement can be represented by a planar quaternion (see Refs. [17,18]). Planar quaternions have been used for designing planar open and closed chains [19,43–45]. Quaternions [46,47,17] are known to have solved the problem of representation of rigid body displacements effectively. They are compact, computationally and storagewise most efficient [48], coordinate frame independent, and unambiguous up to a sign. They also carry a clear geometric meaning, produce singularity free transformations, and concatenate and interpolate nicely.

For a planar displacement shown in Fig. 3, let d_1, d_2 denote the coordinates of the origin of the moving frame $\mathbf{M}$ in the fixed frame $\mathbf{F}$, and α denote the rotation angle of $\mathbf{M}$ relative to $\mathbf{F}$. Then a planar displacement can be represented by a planar quaternion $\mathbf{Z} = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 + Z_3\mathbf{e}_3 + Z_4$, where $(\mathbf{i}, \mathbf{j}, \mathbf{k}, 1)$ forms the quaternion basis and ϵ is the dual unit with the property $\epsilon^2 = 0$. The components of the planar quaternion $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ are given by

$$\begin{aligned} Z_1 &= (d_1/2)\cos(\alpha/2) + (d_2/2)\sin(\alpha/2) \\ Z_2 &= -(d_1/2)\sin(\alpha/2) + (d_2/2)\cos(\alpha/2) \\ Z_3 &= \sin(\alpha/2) \\ Z_4 &= \cos(\alpha/2) \end{aligned} \quad (1)$$

These four components can be identified as coordinates of a point in a four dimensional space. In view of Eq. (1), the coordinates must satisfy the equation

$$Z_3^2 + Z_4^2 = 1 \quad (2)$$

The above equation may be interpreted as defining a hypercircular cylinder in a 4D space.

The components of a planar quaternion are related to the homogeneous transform of a planar displacement by

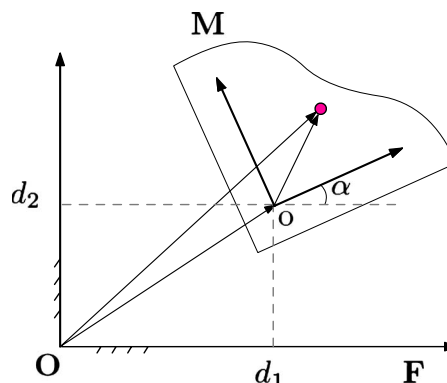


Fig. 3 A planar displacement

$$[A] = \frac{1}{Z_3^2 + Z_4^2} \begin{bmatrix} Z_4^2 - Z_3^2 & -2Z_3Z_4 & 2(Z_1Z_4 - Z_2Z_3) \\ 2Z_3Z_4 & Z_4^2 - Z_3^2 & 2(Z_1Z_3 + Z_2Z_4) \\ 0 & 0 & Z_3^2 + Z_4^2 \end{bmatrix} \quad (3)$$

Note that when $Z_i (i=1, 2, 3, 4)$ is replaced with wZ_i , where w is a nonzero scalar, the matrix $[A]$ is unchanged. From this perspective, the four components of a planar quaternion can also be considered as a set of homogeneous coordinates for a planar displacement. Ravani and Roth [20] considered the components of $\mathbf{Z}$ as defining a point in a projective three-space, called the *image space* of planar displacements, while the point $\mathbf{Z}$ is called the *image point* of a planar displacement.

2.2 Unconstrained Planar B-Spline Motion. This section reviews the idea of synthesis of unconstrained (also known as free-form) planar rational motion. For the purpose of this paper, if the input motion is specified in terms of the given coupler displacements, a rational motion that interpolates the given displacements can be constructed using the method outlined in this section.

Since a planar quaternion $\mathbf{Z}$ can be seen as a point in the image space, a polynomial curve in the image space corresponds to a rational motion in the Cartesian space. Rational motions are defined as the ratio of polynomial functions of time. By applying computer aided geometric design (CAGD) techniques for designing curves in the image space, we obtain rational motions in the Cartesian space. The rationale behind the choice of rational motions stems from the fact that the trajectories of an object undergoing rational motions are rational curves and surfaces, thereby making them convenient for integration into existing nonuniform rational B-spline (NURBS) based computer aided design/computer aided manufacturing (CAD/CAM) system.

Let $\mathbf{Z}_i, i=0, \dots, n$ be given planar quaternions representing displacements, then the following represents a B-spline image curve in the space of planar quaternions:

$$\mathbf{Z}(u) = \sum_{i=0}^n N_{i,p}(u) \mathbf{Z}_i \quad (4)$$

where $N_{i,p}(u)$ are p th-degree basis functions. See Refs. [49–51] for details on the B-spline curves.

A representation for the rational B-spline motion in the Cartesian space can be obtained by substituting the coordinates of $\mathbf{Z}(u)$ into the matrix $[A]$ (Eq. (3)), where each element of the matrix is now obtained as a rational function in parameter u .

3 Constraint Manifold for Planar Chains

In this section, we review the constraint manifold associated with the kinematic constraints of planar 6R closed chains (see Refs. [18,42] for details). The kinematic constraints specify the positions and orientations obtainable by a certain link of the chain. Consider a planar 6R closed chain as shown in Fig. 4. In the figure, $\mathbf{F}$ and $\mathbf{M}$ mark the fixed and the moving frames, respectively. The fixed pivots A_1 and A_2 are located at (x_1, y_1) and (x_2, y_2) , respectively, while the moving frame is located at a distance of h_1 and h_2 from the two end pivots C_1 and C_2 , respectively. The moving frame is assumed to be tilted by angles of α_1 from the line joining the end pivot C_1 and the origin of the moving frame, and α_2 from the line joining the end pivot C_2 and the origin of the moving frame. The length of the links is given by a_1, b_1, a_2 , and b_2 . A planar 6R closed chain can be seen as two 3R open chains (henceforth called left and right open chains) joined together at the ends. Then, the constraint manifold for the planar 6R closed chain is the intersection of constraint manifold of the two 3R open chains. If the displacement of the moving object attached to the moving frame is represented by a planar quaternion $\mathbf{Z}=(Z_1, Z_2, Z_3, Z_4)$, then after eliminating the joint angles from the forward kinematics of each open chain, the algebraic equations for the two manifolds are given by the following.

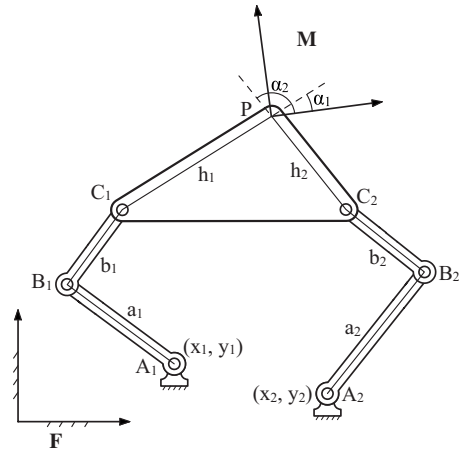


Fig. 4 A planar 6R closed chain

For the left 3R open chain,

$$\frac{(a_1 - b_1)^2}{4} \leq F_1(Z_1, Z_2, Z_3, Z_4) \leq \frac{(a_1 + b_1)^2}{4} \quad (5)$$

where

$$F_1(Z_1, Z_2, Z_3, Z_4) = \frac{(Z_1 - \sigma_1 Z_3 - \tau_1 Z_4)^2 + (Z_2 - \sigma_2 Z_3 - \tau_2 Z_4)^2}{Z_3^2 + Z_4^2} \quad (6)$$

and

$$\begin{aligned} \sigma_1 &= (y_1 + h_1 \sin \alpha_1)/2, & \tau_1 &= (x_1 + h_1 \cos \alpha_1)/2 \\ \sigma_2 &= (-x_1 + h_1 \cos \alpha_1)/2, & \tau_2 &= (y_1 - h_1 \sin \alpha_1)/2 \end{aligned} \quad (7)$$

For the right 3R open chain,

$$\frac{(a_2 - b_2)^2}{4} \leq F_2(Z_1, Z_2, Z_3, Z_4) \leq \frac{(a_2 + b_2)^2}{4} \quad (8)$$

where

$$F_2(Z_1, Z_2, Z_3, Z_4) = \frac{(Z_1 - \xi_1 Z_3 - \eta_1 Z_4)^2 + (Z_2 - \xi_2 Z_3 - \eta_2 Z_4)^2}{Z_3^2 + Z_4^2} \quad (9)$$

and

$$\begin{aligned} \xi_1 &= (y_2 + h_2 \sin \alpha_2)/2, & \eta_1 &= (x_2 + h_2 \cos \alpha_2)/2 \\ \xi_2 &= (-x_2 + h_2 \cos \alpha_2)/2, & \eta_2 &= (y_2 - h_2 \sin \alpha_2)/2 \end{aligned} \quad (10)$$

Equations (5) and (8) characterize the kinematic constraints of a planar 6R closed chain, and define the constraint manifold for the chain.

To visualize the hypergeometric shape described by Eq. (5) or Eq. (8), we project it on the hyperplane $Z_4=1$. Denote $(z_1, z_2, z_3, 1)$ as the projected point of (Z_1, Z_2, Z_3, Z_4) , both of which represent the same planar displacement. Then, $F_1(Z_1, Z_2, Z_3, Z_4)$ on $Z_4=1$ is given by

$$F_1(z_1, z_2, z_3, 1) = \frac{(z_1 - \sigma_1 z_3 - \tau_1)^2 + (z_2 - \sigma_2 z_3 - \tau_2)^2}{z_3^2 + 1} \quad (11)$$

The volume described by Eq. (11) along with the limits of its inequalities creates implicit surfaces of (z_1, z_2, z_3) . Setting $F_1(z_1, z_2, z_3, 1)=c$, where c is a constant, and lies in the range $[(a_1 - b_1)^2/4, (a_1 + b_1)^2/4]$, we reorganize Eq. (5) to obtain

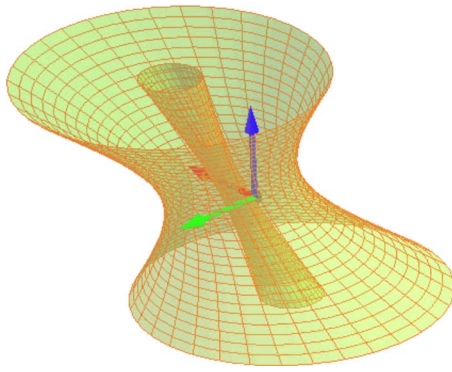


Fig. 5 Visualization of the constraint manifold of a 3R open chain as a pair of concentric, co-oriented, and sheared hyperboloid in the hyperplane $Z_4=1$; two surfaces indicate the limits of the inequality in Eq. (5)

$$\frac{(z_1 - \sigma_1 z_3 - \tau_1)^2}{c} + \frac{(z_2 - \sigma_2 z_3 - \tau_2)^2}{c} - z_3^2 = 1 \quad (12)$$

This is a sheared circular hyperboloid in the space parametrized by (z_1, z_2, z_3) ; see Fig. 5. This hyperboloid has its center at $(\tau_1, \tau_2, 0)$, and the central axis is given by $(z_1 - \tau_1)/\sigma_1 = (z_2 - \tau_2)/\sigma_2 = z_3/1$. Thus, the hyperboloid orients along the vector $(\sigma_1, \sigma_2, 1)$. It is evident that the center and the orientation are dependent on the location of the fixed pivot, the dimensions of the floating link, and the relative orientation of **M** from the floating link. The hyperboloid intersects plane $z_3=0$ in a circle, which has a radius r equal to $\sqrt{c}$. Thus, c determines the size of the hyperboloid. When the value of $F_1(z_1, z_2, z_3, 1)$ varies from its minimum to maximum, the size of the manifold increases correspondingly, but the center point and the orientation remain unchanged. Table 1 summarizes the relationship between geometric features of the hyperboloid and the mechanism parameters. A similar set of relationship exists for the other 3R open chain.

Thus, the constraint manifold of the planar 6R closed chains is given by two pairs of concentric and co-oriented sheared hyperboloid, and for a mechanism to pass through a given motion, the image curve would have to be contained within the constraint manifold.

4 Interactive Dimensional Synthesis

Our design method treats a 6R closed chain as a mechanism assembled using two independent open chains. The constraint manifolds of both the chains are geometric objects in the image space; the size, shape, and position of which are a function of mechanism parameters. A given rational motion maps to an image curve that needs to be contained inside these manifolds. In this section, we describe the procedure for designing planar mechanisms using our tool. We also describe the user interface with which the designer needs to be familiar. The basic idea is that the designers are provided a set of controls via the graphical user interface (GUI) of the tool that will allow them to interactively manipulate the constraint manifold with the objective to contain the image curve in the manifold. Upon being satisfied visually, the

designer will be allowed to instruct the program to check if there are any violations of the kinematic constraints. A windows binary of the tool for x86 architecture can be downloaded.¹ We first present the functionalities of the user interface.

4.1.1 User Interface Functionalities. The GUI has four main parts, as shown in Figs. 1 and 2.

1. *The Cartesian Space Window (CSW).* This window is used to display the given positions and the animation of the open chains in the Cartesian space.
2. *The Image Space Window (ISW).* In this window, the constraint manifold, as well as the image curve projected on the hyperplane $Z_4=1$, is shown.
3. *Motion Design Panel (MoDP).* This panel supports operations such as position insertion, deletion, and modification, and comprises functions to animate the motion and to test for constraint violation. The constraint violation test is done using Eqs. (5) and (8), and the test results are visualized through the user interface. This operation updates both the CSW and the ISW.
4. *Mechanism Design Panel (MeDP).* There are two ways to edit the mechanism: (1) directly manipulate mechanism parameters in the Cartesian space, like the location, the link lengths, and the relative angle, and as a consequence, constraint manifolds change in the image space, or (2) edit the geometric parameters that change the size, position, and the orientation of the manifolds. Designers may find the latter approach more intuitive. For either approach, there are six design variables. Equations (5)–(10) and Table 1 together describe the relationship between mechanism parameters and constraint manifold parameters. We summarize the relationships for one open chain as follows:

$$\begin{aligned} x_1 &= -\sigma_2 + \tau_1 \\ y_1 &= \sigma_1 + \tau_2 \\ h_1 &= \sqrt{\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2} \\ \sin \alpha_1 &= (\sigma_1 - \tau_2) / \sqrt{\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2} \\ \cos \alpha_1 &= (\sigma_2 + \tau_1) / \sqrt{\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2} \\ \{a_1, b_1\} &= \{(r_{\max} + r_{\min}), (r_{\max} - r_{\min})\} \end{aligned} \quad (13)$$

where $r_{\max}$ and $r_{\min}$ describe the outermost and the innermost radius, respectively, of the constraint manifold associated with an open 3R chain. A similar set of relationships exists for the other open chain.

4.1.2 Design Procedure. Now, we present the design procedure.

1. Use the motion design panel to input given positions, associated time parameters, and interpolate them using a NURBS motion.

The given planar positions can be input with the time parameter t , using either planar quaternion coordinates (Z_1, Z_2, Z_3, Z_4) or Cartesian coordinates (x, y, θ) . Once all the given positions are input, a cubic C^2 B-spline motion (Eq. (4)) that interpolates the given positions is generated. Here, the time parameters are input to guarantee the smoothness of the motion. Consequently, the ISW shows the image points of the prescribed positions, and renders a smooth NURBS curve, which passes through all the image points; while the CSW shows the given positions and the rational motion.

Table 1 Parameters for the projective sheared hyperboloid

Geometric features	Constraint parameters
Center	$(\tau_1, \tau_2, 0)$
Orientation	$(\sigma_1, \sigma_2, 1)$
Size	$\frac{\ a_1 - b_1\ }{2} \leq r = \sqrt{c} \leq \frac{a_1 + b_1}{2}$

¹<http://cadcam.eng.sunysb.edu/dimsynth/>

Table 2 Cartesian coordinates of five prescribed positions along with their time parameter values

i	x	y	$\theta(\text{deg})$	u_i
1	2.0448	-0.1940	0	0.0
2	1.9067	1.5029	30	0.2
3	-0.8894	3.4852	-15	0.5
4	-0.7851	3.2652	15	0.5
5	-2.3005	3.1448	31	1.0

- Switch to MeDP. Dimensional synthesis starts with two default 3R open chains. We describe the procedure for one open chain.

In the CSW, initially, the fixed pivots are located at $(x_1, y_1) = (0, 0)$; the three links have unit length $a_1 = b_1 = h_1 = 1$, and the relative angle of $\mathbf{M}$ to the floating link is $\alpha_1 = 0$. With these initial choices and according to Eq. (7) and Table 1, the default hyperboloid pair is centered at $(0.5, 0, 0)$, its direction is parallel to vector $(0, 0.5, 1)$, and the inner boundary circle is of radius $r_{\min} = 0$, while the outer one has a radius $r_{\max} = 1$. Thus, in the ISW, a degenerate inner hyperboloid and a finite sized outer hyperboloid appear. At this point, the image curve may not be completely contained between the two hyperboloids, which means that the constraints are being violated. We note that, in general, initial parameters are difficult to select for body guidance problem solved using optimization approach; however, in our case, this is not an issue because our approach relies on the interactive geometric manipulation of the constraint manifold. Changing the geometric parameters appropriately grows the inner hyperboloid, while shrinks the outer one. This approach provides the designer visual clues for the next step of parametric changes.

- Modify the constraint manifold visually using the spinner controls (up and down arrows next to parameters) provided in the MeDP until the curve seems completely contained within a close fit between the pair of hyperboloids. Dragging the slider in either ISW or CSW verifies if the constraints are actually satisfied or not. Using the current value of the mechanism parameters, the program automatically checks the constraint equations as given by Eqs. (5) and (8) to see if they are satisfied. When they are satisfied, the program outputs links' length, fixed and moving pivot locations, and the orientation of the moving frame.
- Repeat steps 2–4 and synthesize the other open chain.

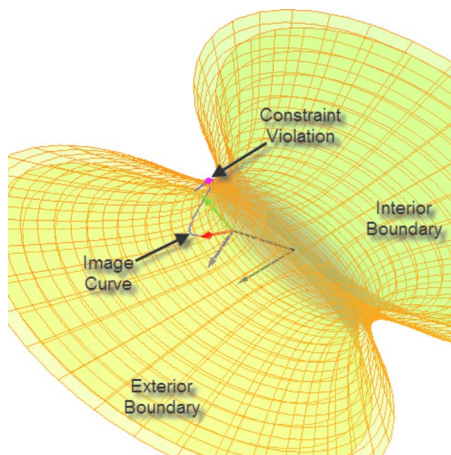


Fig. 6 Constraint manifold of the left 3R open chain and the image curve; in this figure, the image curve is not completely contained inside the manifold. A point of violation is shown.

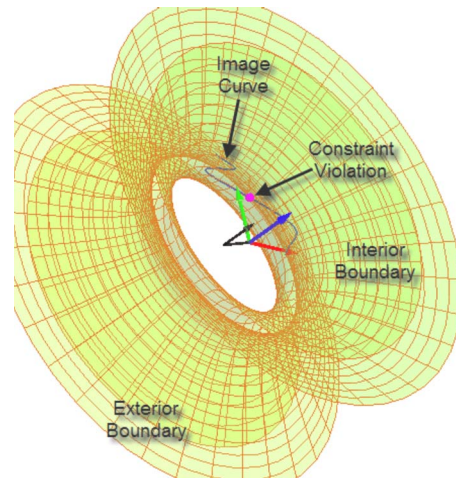


Fig. 7 Constraint manifold of the right 3R open chain and the image curve; in this figure, the image curve is not completely contained inside the manifold. A point of violation is shown.

5 Example

In this section, we present an example that demonstrates how our method interactively completes the dimensional synthesis of a planar 6R closed chain using the constraint manifold modification for a given degree 6 rational motion.

In this example, we use five positions as given in Table 2 and shown in the Cartesian space window in Fig. 1. The positions are given in Cartesian coordinates (x, y, θ) , which specify the location and the orientation of the moving frame $\mathbf{M}$ relative to the fixed frame. Also given are the time parameter values (u_i) associated with each position. We note that time parameters are not required for rigid body guidance problem; we specify them here merely to construct an input B -spline motion. Our method of designing linkage is independent of time parameter values associated with the given positions.

First, the given positions are converted to planar quaternion representation (Z_1, Z_2, Z_3, Z_4) using Eq. (1), and then they are interpolated using a cubic B -spline formulation (Eq. (4)). This gives a degree 6 B -spline rational motion in the Cartesian space. The corresponding image curve is shown in the image space window of Fig. 1. The image curve is visualized using Rodrigues param-

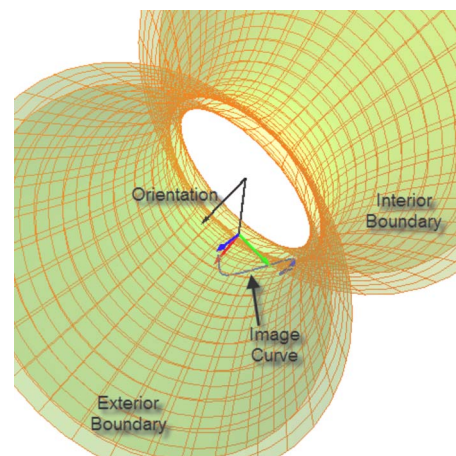


Fig. 8 Constraint manifold of the left 3R open chain and the image curve; in this figure, the image curve is completely contained inside the manifold, thus implying that the constraints are not violated.

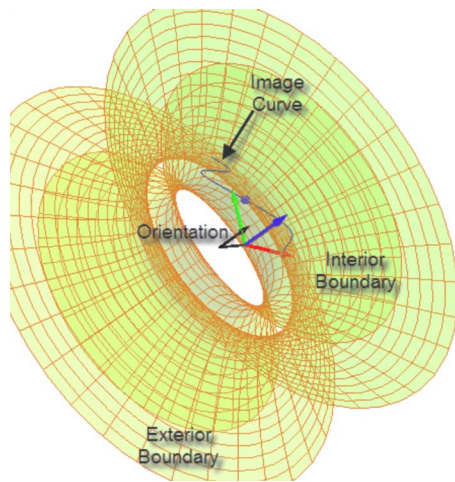


Fig. 9 Constraint manifold of the right 3R open chain and the image curve; in this figure, the image curve is completely contained inside the manifold, thus implying that the constraints are not violated.

eters (see Ref. [17]) given by $(Z_1/Z_4, Z_2/Z_4, Z_3/Z_4)$. Hereafter, the left and right 3R open chains, and their constraint manifolds are initialized, one at a time. However, navigating through the motion, it is found that the constraints are violated—this shows up as the image curve being outside the manifold; see Figs. 6 and 7. The designer next modifies the constraint manifolds interactively by varying its various geometric parameters, such as the size, position, and orientation with the objective to contain the image curve inside the manifolds. The process is intuitive and fast. Once the synthesis of two individual open chains is completed (see Figs. 8 and 9), the assembly of them yields a 6R closed chain that interpolates the given five positions with a smooth motion. Tables 3 and 4 list the design results, and the final mechanism is shown in Fig. 10.

6 Concluding Remarks

In this paper, we presented a simple and intuitive approach to dimensional synthesis of planar 6R closed chains. The approach is based on interactive manipulation of constraint manifold associated with the mechanisms in a 3D environment. The approach is general, and can be extended to other spherical and spatial mechanisms for which constraint manifold are characterized by algebraic equations.

Table 3 Synthesis parameters of the 6R planar closed chain

	x_1	y_1	a_1	b_1	h_1	$\alpha_1(\text{deg})$
Left chain	−1.6	1.7	5.5	0.3	2.9682	122.6
	x_2	y_2	a_2	b_2	h_2	$\alpha_2(\text{deg})$
Right chain	−0.8	−0.22	3.7	0.7	0.8297	164.6

Table 4 Synthesis parameters of the constraint manifolds

	σ_1	σ_2	τ_1	τ_2	$\ a_1 - b_1\ $	$a_1 + b_1$
Left chain	1.6	0	−1.6	−0.9	2.6	2.9
	ξ_1	ξ_2	η_1	η_2	$\ a_2 - b_2\ $	$a_2 + b_2$
Right chain	0	0	−0.8	−0.22	1.5	2.2

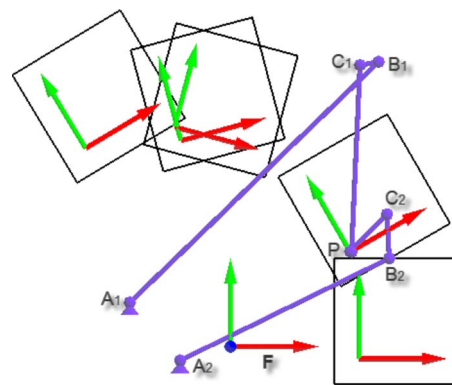


Fig. 10 The final assembled mechanism

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References

- [1] Sandor, G. N., and Erdman, A. G., 1997, *Advanced Mechanism Design: Analysis and Synthesis*, Vol. 2, Prentice-Hall, Englewood Cliffs, NJ.
- [2] Suh, C. H., and Radcliffe, C. W., 1978, *Kinematics and Mechanism Design*, Wiley, New York.
- [3] McCarthy, J. M., 2000, *Geometric Design of Linkages*, Springer-Verlag, New York.
- [4] Rubel, A. J., and Kaufman, R. E., 1977, "Kinsyn III: A New Human-Engineered System for Interactive Computer-Aided Design of Planar Linkages," *ASME J. Eng. Ind.*, **99**, pp. 440–448.
- [5] Erdman, A. G., and Gustafson, J. E., 1981, "Linkages: Linkage Interactive Computer Analysis and Graphically Enhanced Synthesis Packages," *ASME Design Engineering Technical Conferences*, Paper No. 77-DET-5.
- [6] Erdman, A. G., and Riley, D., 1981, "Computer-Aided Linkage Design Using the Linkages Package," *ASME Design Engineering Technical Conferences*, ASME, New York, Paper No. 81-DET-121.
- [7] Kihonge, J., Vance, J., and Larochelle, P., 2001, "Spatial Mechanism Design in Virtual Reality With Networking," *ASME 2001 Design Engineering Technical Conferences*.
- [8] Larochelle, P. M., 1998, "Spades: Software for Synthesizing Spatial 4C Linkages," *ASME Design Engineering Technical Conferences (DETC)*.
- [9] Larochelle, P. M., Dooley, A. P., Murray, A. P., and McCarthy, J. M., 1993, "Sphinx: Software for Synthesizing Spherical 4R Mechanisms," *NSF Design and Manufacturing Systems Conference*, pp. 607–611.
- [10] Ruth, D., and McCarthy, J. M., 1997, "Sphynxpc: An Implementation of Four Position Synthesis for Planar and Spherical 4R Linkages," *ASME Design Engineering Technical Conferences (DETC)*.
- [11] Tse, D., and Larochelle, P. M., 1999, "Osiris: A New Generation Spherical and Spatial Mechanism CAD Program," 1999 Florida Conference on Recent Advancements in Robotics.
- [12] Perez, A., and McCarthy, J. M., 2000, "Dimensional Synthesis of Spatial RR Robots," *Advances in Robot Kinematics*, Kluwer, The Netherlands, pp. 93–102.
- [13] Su, H., and McCarthy, J. M., 2001, "Classification of Designs for RRSS Linkages," *ASME Design Engineering Technical Conferences*.
- [14] Su, H., Collins, C., and McCarthy, J. M., 2002, "An Extensible Java Applet for Spatial Linkage Synthesis," *2002 ASME Design Engineering Technical Conferences*, ASME, New York.
- [15] SyMech, <http://www.symech.com/>.
- [16] WATT, <http://www.heron-technologies.com>.
- [17] Bottema, O., and Roth, B., 1979, *Theoretical Kinematics*, North-Holland, Amsterdam.
- [18] McCarthy, J. M., 1990, *Introduction to Theoretical Kinematics*, MIT, Cambridge, MA.
- [19] Ravani, B., and Roth, B., 1983, "Motion Synthesis Using Kinematic Mappings," *ASME J. Mech., Transm., Autom. Des.*, **105**(3), pp. 460–467.
- [20] Ravani, B., and Roth, B., 1984, "Mappings of Spatial Kinematics," *ASME J. Mech., Transm., Autom. Des.*, **106**(3), pp. 341–347.
- [21] Ge, Q. J., and Ravani, B., 1994, "Computer-Aided Geometric Design of Motion Interpolants," *ASME J. Mech. Des.*, **116**(3), pp. 756–762.
- [22] Ge, Q. J., and Ravani, B., 1994, "Geometric Construction of Bezier Motions," *ASME J. Mech. Des.*, **116**(3), pp. 749–755.
- [23] Jüttler, B., and Wagner, M. G., 1996, "Computer-Aided Design With Spatial

- Rational B-Spline Motions," ASME J. Mech. Des., **118**(2), pp. 193–201.
- [24] Wagner, M. G., 1994, "A Geometric Approach to Motion Design," Ph.D. thesis, Technische Universität, Wien.
- [25] Röschel, O., 1998, "Rational Motion Design—A Survey," Comput.-Aided Des., **30**(3), pp. 169–178.
- [26] Purwar, A., Chi, X., and Ge, Q. J., 2008, "Automatic Fairing of Two-Parameter Rational B-Spline Motion," ASME J. Mech. Des., **130**(1), p. 011003.
- [27] Purwar, A., and Ge, Q. J., 2005, "On the Effect of Dual Weights in Computer Aided Design of Rational Motions," ASME J. Mech. Des., **127**(5), pp. 967–972.
- [28] Blaschke, W., 1911, "Euklidische kinematik und nichteuklidische geometrie," Zeitschr. Math. Phys., **60**, pp. 61–91.
- [29] Grunwald, J., 1911, "Ein abbildungsprinzip, welches die ebene geometrie und kinematik mit der raumlichen geometrie verknüpft," Sitzber. Ak. Wiss. Wien, **120**, pp. 677–741.
- [30] Bodduluri, R. M. C., and McCarthy, J. M., 1992, "Finite Position Synthesis Using the Image Curve of a Spherical Four-Bar Motion," ASME J. Mech. Des., **114**(1), pp. 55–60.
- [31] Bodduluri, R., 1990, "Design and Planned Movement of Multi-Degree of Freedom Spatial Mechanisms," Ph.D. thesis, Univeristy of California, Irving.
- [32] Larochelle, P., 1994, "Design of Cooperating Robots and Spatial Mechanisms," Ph.D. thesis, Univeristy of California, Irving.
- [33] Burmester, L. E. H., 1888, *Lehrbuch der kinematik*, A. Felix, Leipzig.
- [34] Hayes, M. J. D., Luu, T., and Chang, X.-W., 2004, "Kinematic Mapping Application to Approximate Type and Dimension Synthesis of Planar Mechanisms," 9th *Advances in Robotic Kinematics*, Kluwer, Dordrecht.
- [35] Brunthaler, K., Schrocker, H., and Husty, M., 2006, "Synthesis of Spherical Four-Bar Mechanisms Using Spherical Kinematic Mapping," *Advances in Robot Kinematics*, Springer, The Netherlands.
- [36] Venkataramanujam, V., and Larochelle, P., 2007, "Approximate Motion Synthesis of Spherical Kinematic Chains," Proceedings of the ASME 2007 International Design Engineering Technical Conferences and Computers and Information in Engineering, Paper No. DETC2007-34372.
- [37] Jin, Z., and Ge, Q. J., 2007, "Computer Aided Synthesis of Piecewise Rational Motion for Planar 2R and 3R Robot Arms," ASME J. Mech. Des., **129**(10), pp. 1031–1036.
- [38] Jin, Z., and Ge, Q. J., 2007, "Rational Motion Interpolation Under Kinematic Constraints of Planar 6R Closed Chain," ASME 2007 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, ASME, New York, Paper No. DETC2006-99650.
- [39] Purwar, A., Zhe, J., and Ge, Q. J., 2008, "Computer Aided Synthesis of Piecewise Rational Motions for Spherical 2R and 3R Robot Arms," ASME J. Mech. Des., **130**(11), p. 112301.
- [40] Purwar, A., Jin, Z., and Ge, Q. J., 2008, "Rational Motion Interpolation Under Kinematic Constraints of Spherical 6R Closed Chains," ASME J. Mech. Des., **130**(6), p. 062301.
- [41] Purwar, A., Jin, Z., and Ge, Q. J., 2008, "Computer Aided Synthesis of Rational Motion Under Kinematic Constraints of Spatial SS Open Chains," ASME 2008 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference.
- [42] Ge, Q. J., 1990, "Kinematics Constraints as Algebraic Manifolds in the Clifford Algebra of Projective Three Space," Ph.D. thesis, Univeristy of California, Irving.
- [43] Larochelle, P., and McCarthy, J. M., 1994, "Design of Spatial 4C Mechanisms for Rigid Body Guidance," Proceedings of the 1994 ASME Mechanisms Conference, pp. 135–142.
- [44] Murray, A. P., Pierrot, F., Dauchez, P., and McCarthy, J. M., 1997, "A Planar Quaternion Approach to the Kinematic Synthesis of a Parallel Manipulator," *Robotica*, **15**, pp. 361–365.
- [45] Perez, A., and McCarthy, J. M., 2004, "Dual Quaternion Synthesis of Constrained Robotic Systems," ASME J. Mech. Des., **126**(3), pp. 425–435.
- [46] Hamilton, W. R., 1899, *Elements of Quaternions*, Vols. 1–2, Longmans, Green & Co., New York.
- [47] Hamilton, W. R., 1853, *Lectures on Quaternions*, Hodges Smith & Co., Dublin, Ireland.
- [48] Eberly, D., 2002, "Rotation Representations and Performance Issue," Magic Software, Inc., Chapel Hill, NC.
- [49] Farin, G., 1996, *Curves and Surfaces for Computer-Aided Geometric Design: A Practical Guide*, 4th ed., Academic, New York.
- [50] Hoschek, J., and Lasser, D., 1993, *Fundamentals of Computer Aided Geometric Design*, A K Peters, Wellesley, MA.
- [51] Piegl, L., and Tiller, W., 1995, *The NURBS Book*, Springer, Berlin.