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A Task-Driven Unified Synthesis of Planar Four-Bar and Six-Bar Linkages With R- and P-Joints for Five-Position Realization

This paper deals with the problem of integrated joint type and dimensional synthesis of planar four-bar and six-bar linkages, which could contain both revolute (R) and prismatic (P) joints, for guiding through five specified task positions of the end-effector. In a recent work, we developed a simple algorithm for analyzing a set of given task positions to determine all feasible planar dyads with revolute and/or prismatic joints that can be used to guide through the given positions. This paper extends this algorithm to the integrated joint type and dimensional synthesis of Watt I and II and Stephenson I, II, and III six-bar linkages that contain both R- and P-joints. In the process, we developed a new classification for planar six-bar linkages according to whether the end-effector can be constrained by two dyads (type I), one dyad (type II), or no dyad (type III). In the end, we demonstrate this task-driven synthesis approach with three examples including a novel six-bar linkage for lifting an individual with age disability from seating position to standing position. [DOI: 10.1115/1.4033434]

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1 Introduction

This paper deals with the classical problem of synthesizing planar four-bar and six-bar linkages for motion generation, which is also known as rigid-body guidance. A planar four-bar linkage, formed by four binary links with four one-degree-of-freedom (DOF) joints, is the simplest 1DOF closed planar kinematic chain. The 1DOF joint is either a revolute (R) joint or a prismatic (P) joint. The synthesis of a planar four-bar with four revolute joints (or 4R) for motion generation with five task poses, known as the classical Burmester problem, is equivalent to the synthesis of RR dyads. This is a well-studied subject [1,2]. Recently, a comprehensive solution to the Burmester problem that includes not only RR dyads but also PR, RP, and PP has been developed by Angeles and his coworkers [3–6].

The second simplest type of planar closed chain is the six-bar linkages, each of which is formed by four binary links and two ternary links using seven 1DOF joints. When the two ternary links are directly connected via a 1DOF joint, one obtains a Watt six-bar linkage; when two ternary links are connected through a binary link, one obtains a Stephenson six-bar linkage. Depending on the choice of the ground link, there are two distinct types of Watt six-bar linkage (I and II) and three distinct types of Stephenson six-bar linkages (I, II, and III). As the end-effector (or coupler) in a six-bar linkage may be considered as constrained by both dyad (2R) and triad (3R), a six-bar linkage may be designed by developing a triad synthesis method, in addition to dyad synthesis. Chase et al. [7] developed a triad synthesis method for the design of a Stephenson III six-bar linkage for five positions. A homotopy-based approach to synthesize Stephenson six-bar linkages has been reported by Schreiber et al. [8] to deal with additional side conditions. Lin and Erdman [9] were able to design planar triads for six positions. Built on this 3R formulation, Soh and McCarthy [10] presented a procedure for five-position synthesis for the Watt I and Stephenson I, II, and III six-bar structures. Their basic idea is to obtain a 3R chain that guides a moving body through the five given tasks using inverse kinematics. To convert a four-link 3R chain (with the ground link as the fourth link) into a six-bar linkage, they add two 2R dyads to the 3R chain to reduce the DOF of the system from three to one. They noted that this approach does not apply to the Watt II because its floating link is not connected to the ground frame by a 3R chain. Some rectified approaches to synthesis six-bar linkages for four given positions have also been proposed by Bawab et al. [11] as well as Mirth and Chase [12], which seek to eliminate branch and/or circuit defects in addition to the realization of four given positions.

The main contribution of this work is a five-position synthesis algorithm that is applicable to all six topologically different planar linkages consisting of one four-bar and five six-bar structures. In addition, in this paper, we further highlight the advantage of unified treatment for both R-joints and P-joints, i.e., the dyads include RR, RP, and PR and triads include RRR, RRP, RPR, and PRR. This paper builds on and refines our previous conference paper [13], and discussion on dealing with circuit/assembly mode defect has been added (Sec. 3.4), as well as the implementation of a new application on rehabilitation mechanism design based on our approach. The proposed approach is inspired by the results of Soh and McCarthy [10]. Instead of starting the design process with a 3R chain as in their work, we start by analyzing the five task poses first to determine if there exist feasible dyads (RR, RP, PR, and PP) using a planar quaternion-based formulation. According to the number of real solutions in the initial analysis of five task poses, three situations might occur: (a) four real solutions yielding four feasible dyads and thus six feasible four-bar linkages, (b) two real solutions yielding two feasible dyads and thus one feasible four-bar linkage, or (c) no real solution yielding no feasible four-bar linkage. If a feasible four-bar exists and is deemed satisfactory, then no further action is required. If no feasible four-bar exists or none of the feasible four-bar linkages is deemed satisfactory, then we move on to design six-bar linkages for the five task poses.

In synthesizing six-bar linkages, we developed a novel classification of six-bar linkages suitable for task-driven design. It is based on the number of dyads used to constrain an end-effector, which is required to guide through five specified task positions.

For a type I six-bar linkage, the end-effector is constrained by two dyads, which form a four-bar linkage. In this case, the four-bar linkage can be expanded into a six-bar linkage by adding a

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new dyad in two distinct ways. One is to add a triad to either the input link or the output link, and the resulting six-bar linkage is known as a Watt II linkage. The other way is to add a triad to the end-effector, and the resulting six-bar linkage is known as a Stephenson IIIb linkage. In both cases, the addition of a new dyad does not change the number of DOF of the system, which is one (Fig. 1).

For a type II six-bar linkage, the end-effector is constrained only by one dyad. In this case, we expand the 2DOF dyad into a 2DOF five-bar closed chain by adding a triad, again without changing the number of DOF of the system. We then add a binary link to the five-bar chain to reduce the DOF from two to one. This approach leads to four Stephenson six-bar structures (IIa, I, IIb, and III) as shown in Fig. 2.

For a type III six-bar linkage, the end-effector is not constrained by any dyad. In this case, we first make the end-effector to be the end link of a triad so that its DOF remains to be three. We then add a binary link and a ternary link to reduce DOF of the system from three to one. This results in two Watt six-bar structures (Ia and Ib) as shown in Fig. 3.

Essential to the above unified procedure is the algorithm for analyzing five specified task positions for dyad synthesis. This is based on a formulation using planar quaternions pioneered by Bottema and Roth [1] as well as Ravani and Roth [14,15] and followed by the authors of Refs. [16–19]. The solution of the problem requires either the use of algebraic geometry [20,21] or nonlinear numerical optimization methods. We developed a novel algorithm that reduces the five-position synthesis problem into that of finding the roots of a quartic equation [22]. This algorithm can handle all four types of planar dyads with or without prismatic joints.

The paper is organized as follows. Section 2 reviews the concept of planar quaternions in so far as necessary for the development of this paper and introduces a simple algorithm that could handle the dyad synthesis problem with both circle and line constraints for five task position. Section 3 presents a task-driven approach to five-position synthesis for designing four-bar and six-bar linkages as well as a discussion of solution rectification against circuit/assembly mode defect. Section 4 presents three examples to illustrate this approach including the rehabilitation mechanism design application of a novel six-bar linkage, for lifting a disabled individual from seating position to standing position.

2 A Fast Kinematic-Mapping Based Algorithm for Dyad Synthesis

A planar displacement, i.e., a planar pose, consists of translation of a point (d_1, d_2) on the moving body as well as rotation of the body by an angle φ. Let M denote a coordinate frame attached to the moving body and F be a fixed reference frame, as in Fig. 4, then kinematic mapping theory could be employed to convert Cartesian space parameters (d_1, d_2, φ) to image space coordinates $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ (see Ref. [14])

$$\begin{aligned}
Z_1 &= \frac{1}{2}\left(d_1 \sin\frac{\varphi}{2} - d_2 \cos\frac{\varphi}{2}\right)\\
Z_2 &= \frac{1}{2}\left(d_1 \cos\frac{\varphi}{2} + d_2 \sin\frac{\varphi}{2}\right)\\
Z_3 &= \sin\frac{\varphi}{2}\\
Z_4 &= \cos\frac{\varphi}{2}
\end{aligned} \tag{1}$$

The components (Z_1, Z_2, Z_3, Z_4) given above define a planar quaternion that is said to define a point in a projective three-dimensional space called the *image space* of planar displacement, denoted as Σ. In this way, a planar displacement is represented by a point in Σ; a single DOF motion is represented by a curve and a 2DOF motion is represented by a surface in Σ.

Let $\mathbf{Z}_i$ $(i = 1, 2, \ldots, 5)$ denote a planar quaternion associated with five specified task positions of a rigid body. In our previous work [22], we have established a fast and simple algorithm for N planar positions to be approximated by one of the three types of planar dyads: RR, PR, or RP. First, an $N \times 8$ matrix $[A]$ is constructed

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ \vdots & & & & & & & \vdots \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & & & \vdots \\ A_{N1} & A_{N2} & A_{N3} & A_{N4} & A_{N5} & A_{N6} & A_{N7} & A_{N8} \end{bmatrix} \tag{2}$$

The ith row of matrix $[A]$ is consisted by the ith position represented in quaternion form

$$\begin{aligned}
&A_{i1} = Z_{i1}^2 + Z_{i2}^2, A_{i2} = Z_{i1}Z_{i3} - Z_{i2}Z_{i4}\\
&A_{i3} = Z_{i2}Z_{i3} + Z_{i1}Z_{i4}, A_{i4} = Z_{i1}Z_{i3} + Z_{i2}Z_{i4}\\
&A_{i5} = Z_{i2}Z_{i3} - Z_{i1}Z_{i4}, \quad A_{i6} = Z_{i3}Z_{i4}\\
&A_{i7} = Z_{i3}^2 - Z_{i4}^2, \quad A_{i8} = Z_{i3}^2 + Z_{i4}^2
\end{aligned} \tag{3}$$

According to the kinematic mapping theory, for a planar position $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4)$ to be viewed as the moving frame of a planar dyad, it has to satisfy the following quadratic equation:

$$\begin{aligned}
&p_1(Z_1^2 + Z_2^2) + p_2(Z_1Z_3 - Z_2Z_4) + p_3(Z_2Z_3 + Z_1Z_4)\\
&+p_4(Z_1Z_3 + Z_2Z_4) + p_5(Z_2Z_3 - Z_1Z_4) + p_6Z_3Z_4\\
&+p_7(Z_3^2 - Z_4^2) + p_8(Z_3^2 + Z_4^2) = 0
\end{aligned} \tag{4}$$

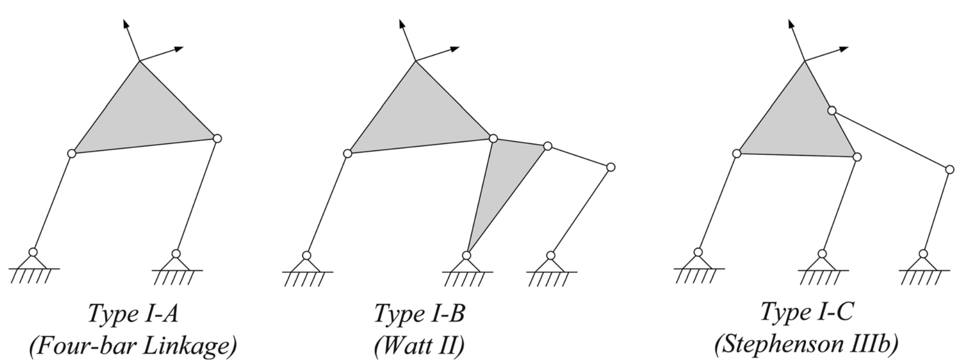

Fig. 1 Three types of linkages constrained by two dyads

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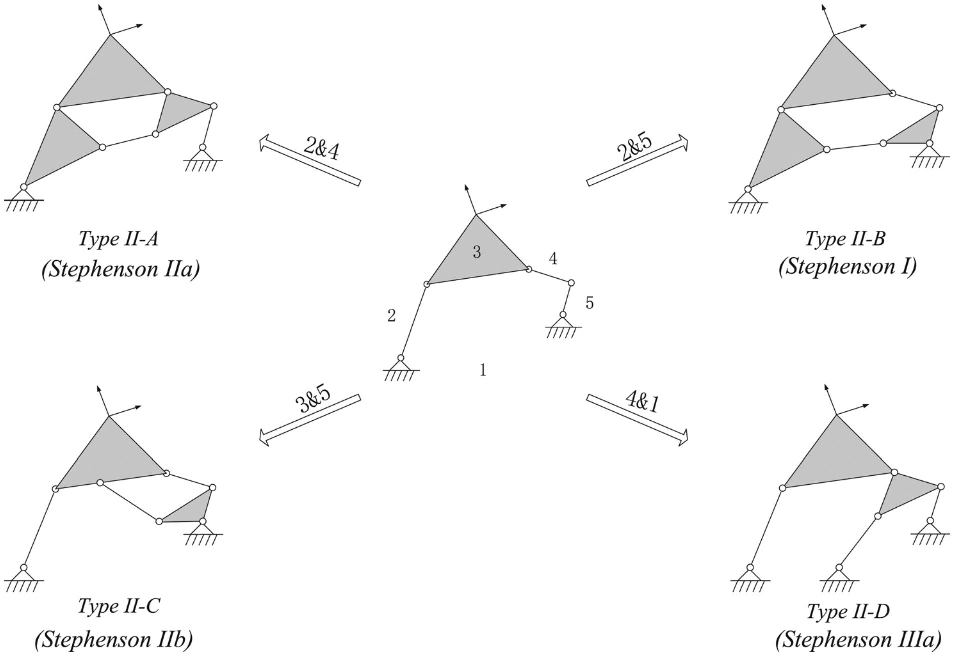

Fig. 2 Four types of Stephenson six-bar linkages. The end-effector (link 3) is constrained by one dyad.

with two additional conditions

$$\begin{aligned} p_1p_6 + p_2p_5 - p_3p_4 &= 0 \\ 2p_1p_7 - p_2p_4 - p_3p_5 &= 0 \end{aligned} \tag{5}$$

where the coefficient vector $\mathbf{p}$ corresponds to one dyad, and the value of each element of $\mathbf{p}$ is then utilized to determine the type and the dimension of the dyad. In this way, we can formulate the five-position motion realization with planar dyads as solving a linear algebraic fitting system

$$[A]\mathbf{p} = 0 \tag{6}$$

with two additional quadratic constraints indicated above in Eq. (5). A solution p_i of the linear system (6) is called a *candidate* solution for the five-position Burmester problem. Only those p_i that also satisfy Eq. (5) correspond to a planar dyad. The candidate solutions are in the null space of $[A]$. Here, we use the singular value decomposition (SVD) method to find the basis vectors of the null space.

With five given positions, matrix $[A]$ contains five rows, making its rank 5. Thus, the matrix $[A]^{\mathrm{T}}[A]$ has three zero eigenvalues, and the corresponding eigenvectors, $\mathbf{v}_\alpha$, $\mathbf{v}_\beta$, and $\mathbf{v}_\gamma$, define the basis for the null space. Let α, β, and γ denote three real parameters. Then, any vector in the null space is given by

$$\mathbf{p} = \alpha\mathbf{v}_\alpha + \beta\mathbf{v}_\beta + \gamma\mathbf{v}_\gamma \tag{7}$$

To find values for (α, β, γ) that satisfy Eq. (5), we substitute Eq. (7) into Eq. (5) and obtain two homogeneous quadratic equations in (α, β, γ)

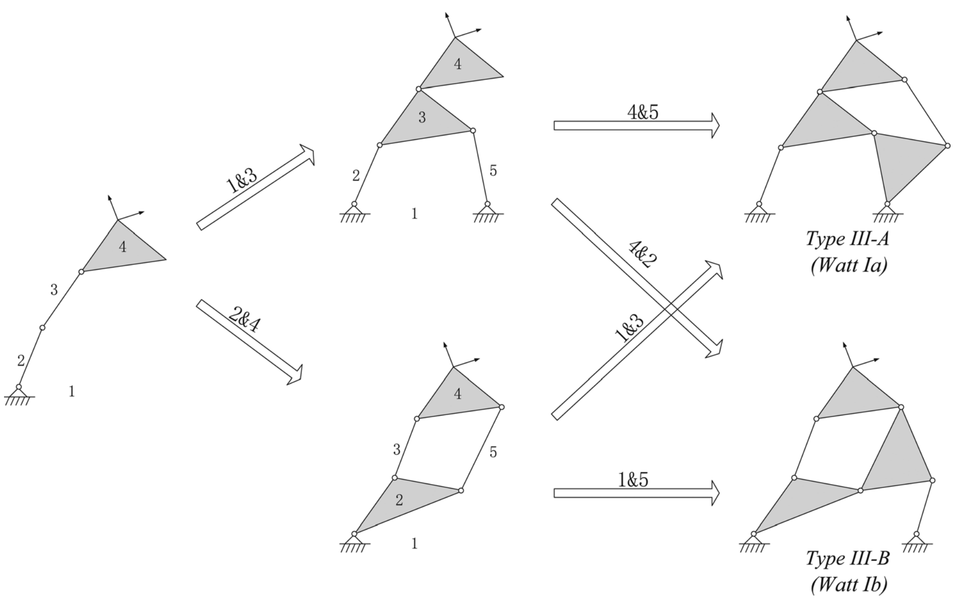

Fig. 3 Two types of Watt I six-bar linkages for each choice of a triad. The end-effector (link 4) is not constrained by any dyad.

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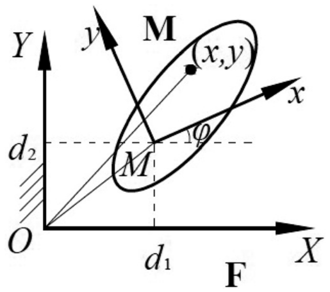

Fig. 4 A planar displacement of a moving frame *M* with respect to the fixed frame *F*

$$K_{10}\alpha^2 + K_{11}\beta^2 + K_{12}\alpha\beta + K_{13}\alpha\gamma + K_{14}\beta\gamma + K_{15}\gamma^2 = 0$$
$$K_{20}\alpha^2 + K_{21}\beta^2 + K_{22}\alpha\beta + K_{23}\alpha\gamma + K_{24}\beta\gamma + K_{25}\gamma^2 = 0 \quad (8)$$

where K_{ij} are defined by components of the three eigenvectors obtained from SVD algorithm. These two quadratic equations can be further reduced to a quartic equation in one unknown in terms of the ratio of two of the three parameters (α, β, γ), and thus can be analytically solved. Since a quartic equation may have four real roots, two real roots, or no real roots, there could be four solutions, two solutions, or no solutions for the coefficients $\mathbf{p}$ of the constraint manifold of planar dyads. As coefficients $\mathbf{p}$ are homogeneous, in this paper, we normalize them such that $\mathbf{p} \cdot \mathbf{p} = 1$.

Furthermore, by investigating whether the solution $\mathbf{p}$ falls into one of the following four patterns, we can determine the type of the resulting dyads:

(1) If $p_1 = p_2 = p_3 = 0$, the resulting dyad is a PR dyad.
(2) If $p_1 = p_4 = p_5 = 0$, the resulting dyad is a RP dyad.
(3) If none of the above, the resulting dyad is a RR dyad.

and the dimension information can also be determined by

$$a_0 : a_1 : a_2 : a_3 = p_1 : p_4 : p_5 : \left(4p_8 - \frac{p_1\left(p_6^2 + 4p_7^2\right)}{p_4^2 + p_5^2}\right)$$
$$x = \frac{p_6 p_5 - 2p_7 p_4}{p_5^2 + p_4^2}, y = -\frac{p_6 p_4 + 2p_7 p_5}{p_5^2 + p_4^2} \quad (9)$$

where (x, y) is the coordinates of the point in the moving frame M that traces a circle, and the expression of that circle in the fixed frame is $a_0(X^2 + Y^2) + 2a_1X + 2a_2Y + a_3 = 0$, which is for RR dyad case. When $p_1 = p_2 = p_3 = 0$, we have $a_0 = 0$, and the circle degenerates into a straight line: $2a_1X + 2a_2Y + a_3 = 0$ and we obtain the dimensions for PR dyad.

Or if it is an RP dyad, the dimensions are

$$l_1 : l_2 : l_3 = p_2 : p_3 : 2p_8$$
$$a_0 : a_1 : a_2 = \left(p_2^2 + p_3^2\right) : (q_3q_6 + 2q_2q_7) : 2(-q_2q_6 + 2q_3q_7) \quad (10)$$

where $\mathbf{l} = (l_1, l_2, l_3)$ is the homogenous line coordinates for the straight line in moving frame M, which always passes through a fixed point $(-a_1, -a_2, a_0)$ in fixed frame F.

3 Task-Driven Approach to Five-Position Synthesis for Four- and Six-Bar Linkages

This section presents the essential contribution of this paper: a new and unified algorithm for task-driven design of four- and six-bar linkages with R- and P-joints for guiding a rigid body through five specified task positions.

We start with analyzing the five given positions using the simple algorithm for the general Burmester problem. This algorithm yields not only the number of dyads that are compatible with the five given positions but also the specific types (RR, RP, PR, and RR) of the dyads as well as their dimensions. In this paper, we call these compatible dyads the *feasible* dyads. One of the distinct advantages of our algorithm is that it can resolve the issue of joint type (R or P) as well as linkage dimensions simultaneously from the same analysis procedure. This process could yield either four feasible dyads, or two feasible dyads, or no feasible dyads from the five positions. Among the feasible dyads, a designer may further determine whether they are acceptable based on additional design requirements such as branch defect, crank requirements, restrictions on link lengths as well as the locations of the fixed and moving pivots, etc. The application of these additional requirements may further reduce the number of feasible dyads.

3.1 Type I: The End-Effector is Constrained by Two Dyads.

When there exist two or more feasible dyads, any two dyads can be used to constrain the end-effector (the coupler link) so that its DOF is reduced from three to one. The result is a four-bar linkage whose fixed pivots and moving pivots can be determined using Eqs. (9) and (10). If need to be, one has two options to expand the resulting four-bar linkage into six-bar linkages without changing the number of DOF. One is to attach another triad between the ground link and one of the input or output links. This results in a Watt II six-bar linkage. Another is to attach a triad to the coupler link. This results in a Stephenson IIIb six-bar linkage.

Figure 1 shows three linkages constrained by two dyads. For comparison, we also include the conventional naming scheme for six-bar linkages as was used by Soh and McCarthy [10].

3.2 Type II: The End-Effector is Constrained by One Dyad.

If only one feasible dyad is deemed satisfactory, the end-effector as part of the dyad has 2DOFs. In this case, we first expand the dyad into a five-bar closed chain by adding a triad, without changing the number of DOF, which is two. The triad could be any of the valid triads defined with R- or P-joints such as RRR, RRP, RPR, and PRR. Once a triad has been selected, the joint parameters could be obtained using inverse kinematics from the five given positions. We then seek to add the sixth binary link to the five-bar chain to obtain a six-bar linkage and at the same time to reduce the DOF of the system from two to one. In this case, the choice of the sixth link is not arbitrary but has to be determined from the five specified task positions. There are four distinct ways of attaching this binary link and they are shown in Fig. 2. They result in four Stephenson six-bar linkages (I, IIa, IIb, and IIIa).

For Stephenson IIa, we first compute five relative positions of link 4 with respect to link 2. The SVD algorithm is then applied to these five positions to determine where to attach the sixth binary link in order to form a four-bar linkage using links 2, 3, 4, and 6. In this case, since the five-bar chain provides one feasible dyad, we know there exist feasible solutions as the resulting quartic equation yields feasible solutions in pairs. Thus, one may obtain up to three feasible Stephenson IIa linkages.

For Stephenson IIb, we first compute five positions of link 3 relative to link 5, and then apply the SVD algorithm to determine where to attach the sixth binary link to form a four-bar linkage using links 3–6. Again, one could obtain up to three feasible Stephenson IIb linkages.

Similarly, one can compute five positions of link 5 relative to link 2 and use links 1, 2, 6, and 5 to form a four-bar linkage. This results in up to three feasible Stephenson I linkages.

For the Stephenson IIIa linkage type, one can use links 1, 6, 4, and 5 to form a four-bar linkage and use five positions of link 4 for dyad synthesis. Again, there could be up to three feasible linkages.

In summary, when there is only one feasible dyad, one may obtain up to 12 Stephenson six-bar linkages for each choice of a triad.

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Table 1 Example 1: A set of five task positions that leads to four feasible dyads

	E_1	E_2	E_3	E_4	E_5
Translation	(3.67, 0.645)	(1.98, 1.56)	(2.56, 1.7)	(3.5, 1.14)	(4.5, 0.55)
Rotation (deg)	−77	−56	−35	−38	−63

3.3 Type III: The End-Effector is Not Constrained by Any Dyad. When the end-effector is not constrained by any dyad, it has 3DOFs and can be easily made as the end link of a triad, without changing the number of DOF. Again, we may select any one of the several triad types involving R- and/or P-joints. To reduce the DOF from three to one, we seek to add one binary link and one ternary link to obtain a six-bar linkage. As shown in Fig. 3, this leads to two Watt six-bar structures (Ia and Ib). The process of adding a new link requires the synthesis of a four-bar linkage as discussed earlier. For Watt Ia, this involves synthesizing the four-bar linkage with links 1, 2, 3, and 5 and subsequently, the second four-bar with links 3, 4, 6, and 5. As each four-bar could yield up to three new solutions, we will have up to nine Watt Ia six-bar linkages. Similarly, we can conclude that we will have up to nine Watt Ib six-bar linkages as well.

In summary, when there is no feasible dyad, one may obtain up to 18 Watt I six-bar linkages for each choice of the triad.

3.4 Discussion on Circuit/Assembly Mode of Resulting Linkage. In practical mechanism design applications, aside from the realization of task positions, the circuit/assembly mode should also be considered for linkages containing four-bar closed-loop structure, i.e., it should be investigated that if all five task positions belong to the same circuit/assembly mode.

As discussed in Sec. 3, a planar dyad constraint (such as RR, RP, or PR) is converted to a quadric in the form of Eq. (4), which could be represented by a hyperboloid-shaped (RR) or hyperbolic–paraboloid shaped (RP and PR) quadratic surface in image space (named a "constraint manifold" or "C-manifold" [23]). The coupler motion of a four-bar closed-loop linkage, which is constructed by two planar dyads, could be represented by the intersection curve of two such C-manifolds in image space. Different circuits/assembly modes are illustrated in the image space as separate intersection curves (see Fig. 5). Since the task positions are converted to a set of points in the image space, after investigating the intersection curve(s), it is clear to identify the circuits/assembly modes of the coupler motion [24,25], and hereby determine if all five task positions lie within the same circuit of the four-bar closed-loop structure. As illustrated in Fig. 5, both resulting dyads are RR, and thus, we obtain an RRRR-type four-bar structure. If there is only one intersection curve as in the right figure, it means that there exists only one circuit/assembly mode, i.e., it is a non-Grashof four-bar structure. But if there are two intersection curves as in the left figure, this implies that the resulting structure contains two circuits, and we need to further investigate whether the five image points (task positions) locate on the same intersection. If not, as in the figure, the five task positions are realized separately in two circuits; then, it leads to the circuit defect.

Thus, to avoid the circuit defect of the resulting four-bar or six-bar linkage, during the above design process in Secs. 4.1, 4.2, or 4.3, one needs to make sure that all the task positions lie on the same intersection curve whenever a dyad is added through SVD algorithm and a four-bar closed-loop structure is formed. If not, the synthesized dyad should be identified as infeasible, and it is necessary to reselect the parameters of the triad in Sec. 4.2 or 4.3 so as to obtain a different structure and set of task positions for the following synthesizing procedure.

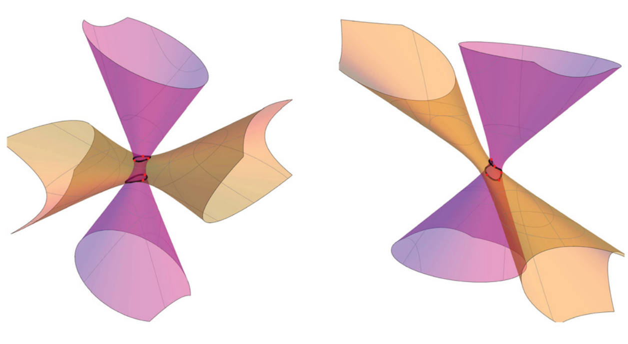

Fig. 5 Coupling of two constraint manifolds (RR open-chain) in image space, where each intersection curve represents one circuit/assembly mode of the resulting RRRR four-bar linkage, and the task positions are converted to a set of image points

4 Examples

In this section, we present three examples to demonstrate the task-driven design process for synthesizing planar linkages that guide through five specified task positions.

4.1 Example 1: Five Task Positions With Four Feasible Dyads. First, consider five task positions given in Table 1. The application of our task analysis algorithm leads to four feasible RR dyads. The circle points (x, y) and the homogeneous coordinates of the circle constraints (a_0, a_1, a_2, a_3) are obtained using Eq. (9). The results are listed in Table 2 and shown graphically in Fig. 6. These four circle constraints define four RR dyads, which could be used to construct six four-bar linkages.

4.2 Example 2: Five Task Positions With Two Feasible Dyads. Now, consider another set of five task positions listed in Table 3. In this case, our task analysis algorithm yields only two real solutions $\mathbf{p}_1$ and $\mathbf{p}_2$ for homogeneous coordinates of the constraint manifolds of planar dyads. Listed in Table 4 are the homogeneous coordinates of the two circle constraints and coordinates of two circle points. These two circle constraints define two dyads, which could be used to construct a four-bar linkage.

Suppose only one of the two dyads, say, for example, $\mathbf{p}_2$, is deemed acceptable. We choose to use a triad (3R) to replace the dyad associated with $\mathbf{p}_1$. The pivot on the moving body is then selected to be at (3, 0) and the fixed pivot to be (1, 1). The link lengths of the first two links are set to be identical, which is 2. The location of the second joint can be easily calculated by inverse kinematics: $(0.7488, -0.9842), (1.4125, -0.9570), (1.8163, -0.8258), (2.6789, -0.0867)$, and $(2.8306, 0.1945)$. This results in a five-bar linkage with 2DOFs such that its third link passes through the five given positions. As shown in Fig. 2, we seek to add the sixth link between links 4 and 1 to obtain a Stephenson IIIa linkage. The next step is to use the five positions of link 4 to find one or more dyads that are compatible with the resulting five-bar linkage. As feasible dyads must come in pairs and there is already a feasible one, which is formed by links 4 and 5, we know that there exists at least one, perhaps three feasible solutions to connect the sixth link from the ground link to link 4. In this example, there exists only one solution for the sixth link, which is shown in Fig. 7(*a*).

As stated earlier, one of the important advantages of our approach is the generality in the type of joints. In other words,

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Table 2 Example 1: Coordinates of circle constraints and circle points of the four resulting dyads

Vector	a_0:a_1:a_2:a_3	x	y	Dyad type
$\mathbf{P}_1$	1 : −4.8708 : 1.8109 : 25.3914	3.4815	−0.3537	RR dyad
$\mathbf{P}_2$	1 : 0.8353 : −2.3813 : 4.4677	−3.3112	−2.7609	RR dyad
$\mathbf{P}_3$	1 : −0.2441 : 4.3333 : 6.3577	0.9667	−2.7887	RR dyad
$\mathbf{P}_4$	1 : −5.1255 : −8.1550 : 54.5546	−1.3472	0.5821	RR dyad

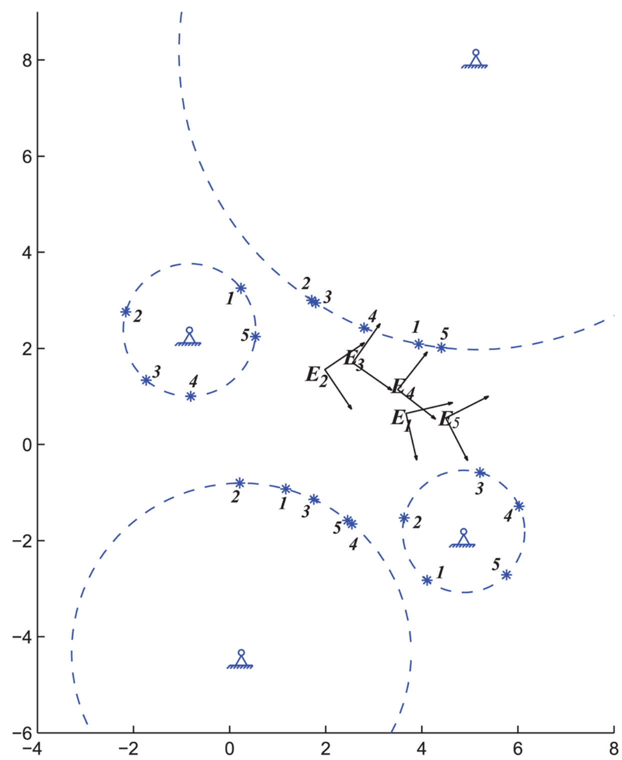

Fig. 6 Example 1: Four circle constraints defining four RR dyads

since our algorithm handles both revolute joint and prismatic joint with the same approach, therefore, the resulting best-fit mechanisms could contain either type of joints. To illustrate this point, we now select a different 3R chain such that the third joint is located at (2, 0), the first joint (the ground joint) is changed to (0, 1), and the two link lengths remain to be 2. It follows that the locations of the second joint are: (0.7362,−0.8596), (1.3691,−0.4580),(1.8097,0.1484),(1.9961,0.8744), and(1.9021, 1.6180). In this case, our algorithm generates three dyads that could be used to define the sixth link between links 4 and 1, two of which are RR dyads and the third one is a PR dyad (Table 5). The first three coordinates of $\mathbf{p}_1$ are close to zero, which means that the resulting RR dyad has such a large radius that it could be approximated by a PR dyad. Shown in Fig. 7(*b*) is the Stephenson IIIa six-bar linkage that includes this P-joint.

This example shows that the type of joints in the resulting mechanism is not determined by preselection but by computation. For theoretical motion synthesis, this indicates that our approach achieves both type synthesis and dimensional synthesis simultaneously with one unified algorithm. In practical machine design cases, our algorithm could yield the optimal linkage without going through all different combinations of joint types.

4.3 Example 3: A Six-Bar Linkage Generating a Sit-to-Stand (STS) Motion.

STS motion executed by individuals is a biomechanically demanding task requiring muscle strength greater than other activities of daily life, such as ambulation or stair climbing [26]. It is also known that more than two million people of age 64 or older in the U.S. have difficulty in rising from a chair [27]. In this section, we are seeking to design a linkage mechanism which is incorporated in a custom wheelchair that enables people with such disability to stand up from a seated position. We specify five task positions sampled from the dorsal motion of a healthy person during STS process:

Table 3 Example 2: A set of five task positions that leads to two feasible dyads

	E_1	E_2	E_3	E_4	E_5
Translation	(−0.8556, 2.0065)	(0.2122, 2.9396)	(2.1209, 4.0810)	(3.9846, 4.3271)	(3.3990, 5.1619)
Rotation (deg)	−26	−45	−85	−125	−97

Table 4 Example 2: Coordinates of circle constraints and circle points of the two resulting dyads

Vector	a_0: a_1: a_2: a_3	x	y	Dyad type
$\mathbf{P}_1$	1 : 1.0109 : −5.5980 : 8.6605	2.0255	0.0430	RR dyad
$\mathbf{P}_2$	1 : −2.7224 : −1.1771 : 8.4095	3.5505	0.2013	RR dyad

Table 5 Example 2: Homogeneous coordinates for three dyads that could be used to define link 6 between links 4 and 1

$\mathbf{P}_1$	0.0019	−0.0040	0.0042	−0.2321	0.4498	0.4359	0.7315	0.1365	PR dyad
$\mathbf{P}_2$	0.0140	−0.0265	0.0349	−0.2720	0.4330	0.1427	0.7954	0.2887	RR dyad
$\mathbf{P}_3$	0.0143	−0.0656	0.0894	−0.1176	−0.0254	−0.8517	0.1902	0.4598	RR dyad

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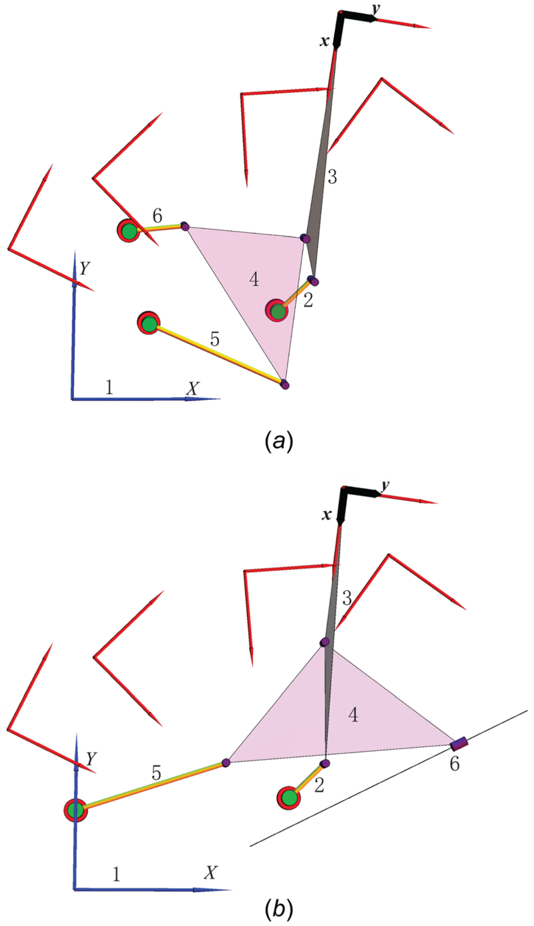

Fig. 7 Example 2: Two six-bar linkages obtained by constraining a five-bar chain with an additional link connecting to the ground: a Stephenson IIIa six-bar linkage with no P-joint (*a*) and one P-joint (*b*)

(6.6, 71.8, 0 deg), (19.46, 78.69, −37 deg), (27.08, 88.85, −30 deg), (32.15, 101.54, −17.5 deg), and (38.15, 108.31, 0 deg) with the origin of the moving frame being at the center of the wheel.

After analysis of the five task positions, it turned out that they yield no feasible dyads, so we start with a 3R chain and synthesize a six-bar linkage. As shown in Fig. 8, the third joint of the 3R

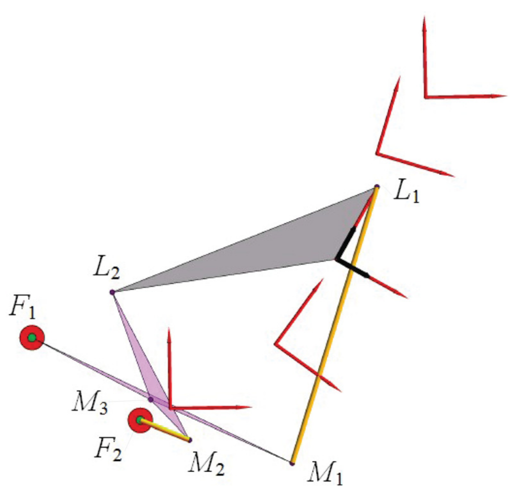

Fig. 8 Example 3: The synthesized six-bar linkage for the generation of STS motion at the third task position

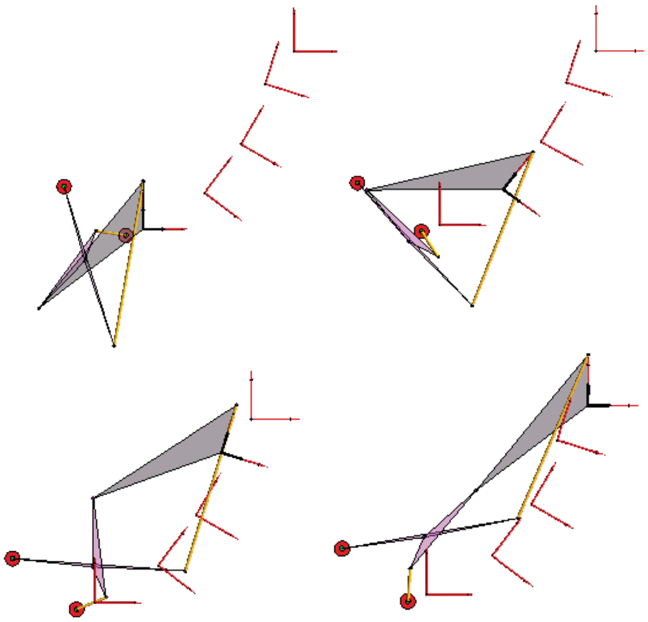

Fig. 9 Example 3: The sit-and-stand six-bar linkage at task positions 1 and 2 (top) and 4 and 5 (bottom)

chain (L_1) is selected at the shoulder, i.e., (0, 10) in the moving frame, and the ground joint (F_1) is at $(-10, 80)$ and the lengths of links 2 (F_1M_1) and 3 (M_1L_1) are both set to be 35. Following the procedure in Fig. 3, we next choose to add link 5 (L_2M_3) between links 2 (F_1M_1) and 4 (L_1L_2). Using our SVD-based motion synthesis algorithm in Sec. 3 to analyze the relative positions of link 4

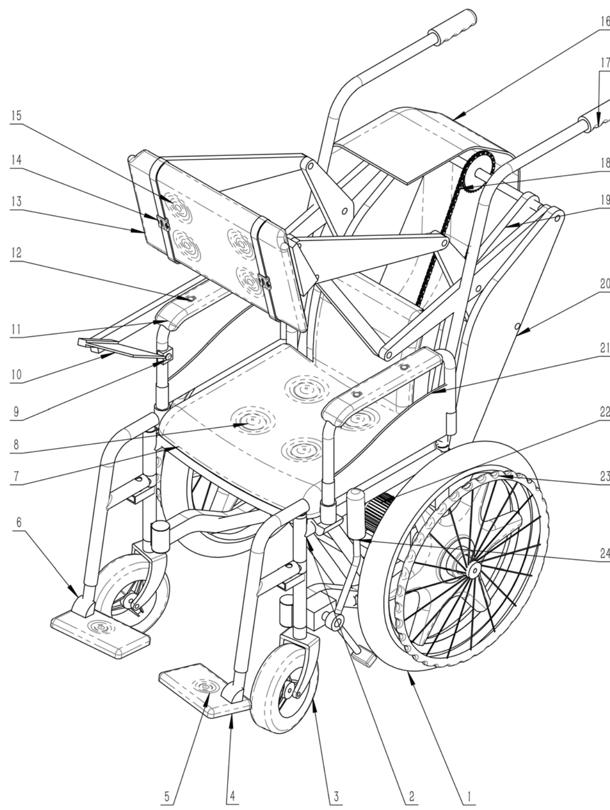

Fig. 10 Example 3: The sketch for the prototype that employs the six-bar linkages designed in Fig. 8, with position 4 (Fig. 9) being realized

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with respect to link 2, we obtain link 5: $L_2 = (-0.1867, 85.3220)$ and $M_3 = (8.8308, 67.2514)$ at the position in Fig. 8. Adding link 5 reduces the DOF of the system from three to two.

Link 6 (F_2M_2) is added between links 5 (L_2M_3) and 1 (ground) to reduce the system DOF from two to one. Similarly as above, after computing the five positions of link 5 with respect to the ground and conduct our SVD-based motion synthesis algorithm, we obtain the locations of two R-joints of link 6: $F_2 = (2.9477, 69.7299)$ and $M_2 = (4.2413, 72.2061)$ in Fig. 8. Figure 9 shows the synthesized Watt Ia six-bar linkage passing through the remaining four specified task positions. Figure 10 illustrates a prototype for the assistive wheelchair, with the resulting six-bar linkage equipped at the seat back part.

5 Conclusions

In this paper, we developed a task-driven unified methodology for synthesizing four-bar and six-bar linkages for five-position motion generation. Central to our methodology is an SVD-based task analysis algorithm that reduces the five-position synthesis problem to the solution of a quartic equation. The algorithm is formulated using planar quaternions which lead to a unified representation of all four types of planar dyads, RR, RP, PR, and PP, using a set of eight homogeneous coordinates $(p_1, p_2, \ldots, p_8)$ satisfying two quadratic equations. A novel feature of this algorithm is that it solves the five-position Burmester problem by *analyzing* the five given positions directly. One can now determine, directly, whether a particular set of five positions can be realized with any of the four planar dyads, and if not, one can proceed to synthesize six-bar linkages for five-position motion generation. In addition, this paper also provides a new classification of planar six-bar linkages based on whether a rigid body (the end-effector) can be constrained with two dyads (type I), or one dyad (type II), or no dyad (type III). While the traditional classification, named after Watt and Stephenson, is based on how four binary links and two ternary links are connected and is more suitable for the analysis of six-bar linkage, our classification focuses on how the end-effector is constrained as part of a six-bar linkage and is thus more suitable for task-driven design.

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References

[1] Bottema, O., and Roth, B., 1979, *Theoretical Kinematics*, North-Holland, Amsterdam, The Netherlands.

[2] McCarthy, J. M., 1990, *Introduction to Theoretical Kinematics*, MIT, Cambridge, MA.

[3] Angeles, J., and Bai, S., 2005, "Some Special Cases of the Burmester Problem for Four and Five Poses," ASME Paper No. DETC2005-84871.

[4] Bourrelle, J., Chen, C., Caro, S., and Angeles, J., 2007, "Graphical User Interface to Solve the Burmester Problem," 12th IFToMM World Congress, Besancon, France, June 17–20.

[5] Chen, C., Bai, S., and Angeles, J., 2008, "A Comprehensive Solution of the Classical Burmester Problem," Trans. CSME, **32**(2), pp. 137–154.

[6] Chen, C., Bai, S., and Angeles, J., 2008, "The Synthesis of Dyads With One Prismatic Joint," ASME J. Mech. Des., **130**(3), p. 034501.

[7] Chase, T. R., Erdman, A. G., and Riley, D. R., 1987, "Triad Synthesis for up to Five Design Positions With Application to the Design of Arbitrary Planar Mechanisms," ASME J. Mech., Transm., Autom. Des., **109**(4), pp. 426–434.

[8] Schreiber, H., Meer, K., and Schmitt, B. J., 2002, "Dimensional Synthesis of Planar Stephenson Mechanisms for Motion Generation Using Circlepoint Search and Homotopy Methods," Mech. Mach. Theory, **37**(7), pp. 717–737.

[9] Lin, C. S., and Erdman, A. G., 1987, "Dimensional Synthesis of Planar Triads for Six Positions," Mech. Mach. Theory, **22**(5), pp. 411–419.

[10] Soh, G.-S., and McCarthy, J., 2008, "The Synthesis of Six-Bar Linkages as Constrained Planar 3R Chains," Mech. Mach. Theory, **43**(2), pp. 160–170.

[11] Bawab, S., Kinzel, G. L., and Waldron, K. J., 1996, "Rectified Synthesis of Six-Bar Mechanisms With Well-Defined Transmission Angles for Four-Position Motion Generation," ASME J. Mech. Des., **118**(3), pp. 377–383.

[12] Mirth, J. A., and Chase, T. R., 1995, "Circuit Rectification for Four Precision Position Synthesis of Stephenson Six-Bar Linkages," ASME J. Mech. Des., **117**(4), pp. 644–646.

[13] Zhao, P., Purwar, A., and Ge, Q. J., 2013, "Task Driven Unified Synthesis of Planar Four-Bar and Six-Bar Linkages With Revolute and Prismatic Joints for Five Position Synthesis," ASME Paper No. DETC2013-13168.

[14] Ravani, B., and Roth, B., 1983, "Motion Synthesis Using Kinematic Mappings," ASME J. Mech., Transm., Autom. Des., **105**(3), pp. 460–467.

[15] Ravani, B., and Roth, B., 1984, "Mappings of Spatial Kinematics," ASME J. Mech., Transm., Autom. Des., **106**(3), pp. 341–347.

[16] Larochelle, P., 1996, "Synthesis of Planar RR Dyads by Constraint Manifold Projection," ASME Paper No. DETC/MECH-1187.

[17] Larochelle, P., 2003, "Approximate Motion Synthesis of Open and Closed Chains Via Parametric Constraint Manifold Fitting: Preliminary Results," ASME Paper No. DETC2003/DAC-48814.

[18] Hayes, M. J. D., and Murray, P. J., 2002, "Solving Burmester Problem Using Kinematic Mapping," ASME Paper No. DETC/CIE2002/DAC-1234.

[19] Hayes, M. J. D., Luu, T., and Chang, X.-W., 2004, "Kinematic Mapping Application to Approximate Type and Dimension Synthesis of Planar Mechanisms," *On Advances in Robot Kinematics*, Springer, Dordrecht, The Netherlands, pp. 41–48.

[20] Brunnthaler, K., Pfurner, M., and Husty, M., 2006, "Synthesis of Planar Four-Bar Mechanisms," Trans. CSME, **30**(2), pp. 297–313.

[21] Husty, M. L., Pfurner, M., Schrocker, H.-P., and Brunnthaler, K., 2007, "Algebraic Methods in Mechanism Analysis and Synthesis," Robotica, **25**, pp. 661–675.

[22] Ge, Q. J., Zhao, P., Purwar, A., and Li, X., 2012, "A Novel Approach to Algebraic Fitting of a Pencil of Quadrics for Planar 4R Motion Synthesis," ASME J. Comput. Inf. Sci. Eng., **12**(4), p. 041003.

[23] Ge, Q. J., Zhao, P., and Purwar, A., 2013, "A Task Driven Approach to Simultaneous Type and Dimensional Synthesis of Planar Four-Bar Linkages Using Algebraic Fitting of a Pencil of Quadrics," ASME Paper No. IDETC2013-12977.

[24] Li, X., Ge, X., Purwar, A., and Ge, Q. J., 2015, "A Unified Algorithm for Analysis and Simulation of Planar Four-Bar Motions Defined With R- and P-Joints," ASME J. Mech. Rob., **7**(1), p. 011014.

[25] Schrocker, H.-P., Husty, M. L., and McCarthy, J. M., 2007, "Kinematic Mapping Based Assembly Mode Evaluation of Planar Four-Bar Mechanisms," ASME J. Mech. Des., **129**(9), pp. 924–929.

[26] Ploutz-Snyder, L. L., Manini, T., Ploutz-Snyder, R. J., and Wolf, D. A., 2002, "Functionally Relevant Thresholds of Quadriceps Femoris Strength," J. Gerontol., Ser. A, **57**(4), pp. B144–B152.

[27] Dawson, D., Hendershot, G., and Fulton, J., 1987, "Aging in the Eighties: Functional Limitations of Individuals Age 65 and Over," National Center for Health Statistics Advance Data, U.S. Department of Health and Human Services, Atlanta, GA, Vol. 133, pp. 1–11.

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