Putting all eggs in one basket: some insights from a correlation inequality

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Abstract

We give examples of situations – stochastic production, military tactics, corporate merger – where it is beneficial to concentrate risk rather than to diversify it, *i.e.*, to put all eggs in one basket. Our examples admit a dual interpretation: as optimal strategies of a single player (the "principal") or, alternatively, as dominant strategies in a non-cooperative game with multiple players (the "agents").

The key mathematical result can be formulated in terms of a convolution structure on the set of increasing functions on a Boolean lattice (the lattice of subsets of a finite set). This generalizes the well-known Harris inequality from statistical physics and discrete mathematics; we give a simple self-contained proof of this result, and prove a *further* generalization based on the game-theoretic approach.

Keywords: Harris inequality, correlation inequality, increasing functions, diversifying risk, concentrating risk, stochastic production, military tactics, corporate merger, dominant strategy.

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Economics Classification – JEL: C60, C61, C72, G10.

Introduction

> "It is the part of a wise man to keep himself today for tomorrow, and not to venture all his eggs in one basket." — M. Cervantes (Don Quixote, 1605)
>
> "Behold, the fool saith, "Put not all thine eggs in the one basket" — which is but a matter of saying, "Scatter your money and your attention"; but the wise man saith, "Put all your eggs in the one basket and – WATCH THAT BASKET."" — M. Twain (Pudd'nhead Wilson, 1894)

Although modern economic theory and practice widely advocate for risk diversification—epitomized by the maxim of distributing 'eggs' across various 'baskets' and embraced alike by academic scholars and by financial experts such as bankers, investors, and portfolio managers—there are circumstances where it may be beneficial to concentrate risk. This is especially true in cases where the potential gains from a single successful venture can eclipse the losses from several failures, and where the magnitude of a possible reward justifies the concentration of all resources into one endeavor. Our analysis aims to illuminate various scenarios where adopting a strategy focused on a single 'basket' is advantageous.

In the realm of portfolio theory, the typical payoff function is the *sum* of returns from various investments, and then it is beneficial to create strategic diversification by bundling together asset classes that exhibit no correlation or negative correlation. In contrast, our study explores scenarios where the payoff is the *product* of exogenously given functions, and is influenced by strategic choices that affect their joint distribution. The pivotal insight is that positive correlation among these functions could make it strategically beneficial to adopt a coordinated approach, which effectively concentrates risk and increases the probability that the peaks and troughs of the individual functions occur together.

We present a series of examples demonstrating that such scenarios can emerge quite naturally in a strategic setting. Our examples span a variety of contexts, including stochastic production with unpredictable input supplies, military tactics aimed at disrupting enemy communication networks, and the decision-making processes in corporate mergers. Each scenario can be interpreted in two ways: either as an optimal strategy for an individual actor (the 'principal') or as dominant strategies in a non-cooperative setting involving several participants (the 'agents').

In each scenario presented, the principal's potential strategies are subsets S of a finite set H. The case where $S = \emptyset$ represents the strategy of maximum risk diversification, while $S = H$ signifies the strategy of maximum risk concentration. We demonstrate that, in every instance, the payoff function $\Pi(S)$ is an increasing function, meaning that if S contains T then $\Pi(S) \geq \Pi(T)$. Consequently, the optimal strategy is to choose $S = H$, effectively 'putting all eggs in one basket'.

The unifying mathematical principle across the three examples is delineated as follows: Let $\mathscr{H}$ be the power set of a finite set H, and let A be the space of real-valued functions on $\mathscr{H}$. In (4) below, we define a convolution operation $f \star g$ on A, predicated on a 'coin tossing' mechanism with probabilities $0 \leq p_h \leq 1$ for each element $h \in H$. Our first main result, Theorem 11, establishes that the convolution of two increasing functions results in another increasing function. Now it turns out that in each example, the payoff function Π can be expressed as the convolution of two increasing functions, confirming that Π itself is an increasing function.

While the initial examples are binary in nature, Section 5 expands the discussion to a more complex scenario of stochastic production with multiple inputs. In this broader context, the principal's strategies correspond to partitions of the input set. We establish that the optimal strategy involves opting for the coarsest partition. To substantiate this, we recast the problem as a strategic game among the agents. Theorem 20 asserts that selecting the coarsest partition constitutes a dominant strategy for each agent—a notion reinforced by Remark 7.4.

Theorem 20 has a further implication. In the context of stochastic production, as detailed in Section 5, it may well happen that the principal does not have the capacity to directly execute her optimal strategy. Instead, she must rely upon each of her autonomous agents to "fall in line", i.e., to voluntarily choose to implement his component of her optimal strategy. Remark 7.5 illustrates that by allocating a modest share of her payoff to each agent, the principal can effectively incentivize them to act in accordance with her strategy. This strategic distribution of incentives ensures that the principal's optimal strategy is self-enforcing!

It is intriguing that Theorem 20 not only generalizes Theorem 11 but also admits a very simple proof. In fact, one might well question the need for a separate proof of Theorem 11. Our rationale is threefold: first, *both* proofs are concise and standalone. Second, Theorem 20, while broader, is more technical; it involves a special class of games with elaborate assumptions on strategies and payoffs. Most importantly, our exposition is crafted to be accessible to both economists and probabilists; hopefully serving as a bridge between the two disciplines.

Indeed, Theorem 11 belongs to the class of correlation inequalities that are widely studied and applied in combinatorics, graph theory, and statistical physics. In particular it readily implies the well-known Harris inequality [8], which a pivotal concept in percolation theory and the Erdős-Rényi model of random graphs [2]. We hope that our paper will serve to introduce the beautiful subject of correlation inequalities to those previously unacquainted with the topic.

The Harris inequality and its extensions, such as the Fortuin-Kasteleyn-Ginibre inequality [7] and the Ahlswede-Daykin four-function inequaliy [1], have been applied in economic theory to several areas including comparative statics [3, 12], bargaining networks [4], and optimal assignments [9]. However their role in risk concentration strategies, which is the central theme of our paper, remains unexplored. While the paper [11] does discuss risk concentration, it is underpinned by a different set of principles, and does not employ correlation inequalities.

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1 Example 1: Stochastic production with two inputs

An entrepreneur has a Cobb-Douglas (log-linear) production function of the form

$$f(x,y) = x^{\alpha} y^{\beta}, \quad \alpha, \beta > 0, \tag{1}$$

involving two inputs x, y, which she sources from a finite set H of suppliers.

Each supplier $h \in H$ has x_h units of the x-input and y_h units of the y-input which he needs to ship to the entrepreneur, either separately by way of two independent shipments, or together in one "pooled" shipment. The shipments have zero dollar cost (see, however, Remark 7.2) but are fraught with risk: the probability that a shipment by h will reach its destination is p_h. These probabilities are independent across the suppliers and also across different shipments by the same supplier.

The entrepreneur can make a *separating contract* with supplier h for him to ship x_h and y_h independently; or a *pooling contract* for him to ship them jointly. She has to give all her suppliers sufficient advance notice and make these contracts *ex ante*, prior to the realisation of any of their deliveries, *i.e.,* the luxury of telling a supplier h what to do, conditional on the deliveries obtained from other suppliers, is not available to her. Thus her possible strategies are indexed by subsets $S \subseteq H$, where the *S-strategy* consists of making a pooling contract with each supplier in S, and a separating contract with all the other suppliers.

Which strategy $S \subseteq H$ will maximize the entrepreneur's expected output?

There are clear-cut advantages of diversifying risk by means of separating contracts. For suppose that many shipments fail under pooling, so that the total x and y received by the entrepreneur are both small. Had she opted for separating contracts, there would have remained a chance of less failure of the y-shipments despite the widespread failure of the x-shipments, enabling the production of medium output. Under pooling, x and y are inexorably linked: if x is small, so is y, and therefore so is the output. Why should the entrepreneur not diversify risk, instead of putting all the eggs (inputs of h) in the same basket (shipment of h)? However we show that she should do just that.

Indeed, consider the special case of a single supplier, with $H = \{1\}$. Then the expected output is $px_1^{\alpha}y_1^{\beta}$, where p is the probability that the entrepreneur receives *both* x and y from the supplier. This probability is p_1 under pooling, and $p_1^2 \leq p_1$ under separate shipments. Thus pooling leads to higher expected output.

We will show that the advantage of pooling persists with many suppliers, and with a more general class of production functions that we now describe.

As before, we consider an entrepreneur who sources two kinds of input, x and y, from a finite set H of suppliers. Each supplier h can ship both kinds of inputs in a single pooled shipment or in two separate shipments, and each shipment by h arrives with probability p_h. The S-strategy of the entrepreneur consists of making a pooling contract with all $h \in S$ and a separating contract with all $h \notin S$.

However we suppose now that the entrepreneur has a more general production function of the form

$$F(S_1, S_2) = F_1(S_1)F_2(S_2), \tag{2}$$

where S_1 and S_2 are the sets of suppliers from whom she receives shipments of x and y, respectively, and F_1 and F_2 are increasing functions.

The S-strategy induces a probability distribution on pairs (S_1, S_2), and thus on the output, and we let $\Pi_1(S)$ denote the expected output.

Claim 1 *The expected output $\Pi_1(S)$ is an increasing function of S.*

Proof. See Section 4. ■

Corollary 2 *The H-strategy maximizes expected output.*

Remark 3 *The Cobb-Douglas function* (1) *is a special case of* (2) *with $F_1(S_1) = (\sum_{h \in S_1} x_h)^{\alpha}$, $F_2(S_2) = (\sum_{h \in S_2} y_h)^{\beta}$. Clearly F_1, F_2 are increasing in S_1, S_2.*

2 Example 2: Military tactics

Country I has two communication networks, R (red) and B (blue). Each network is characterized by a pair $(H_\alpha, \mathscr{C}_\alpha)$, $\alpha \in \{R, B\}$. Here H_α is a finite set of sites across the country at which there exist hubs of network α; and $\mathscr{C}_\alpha$ is a collection of *critical subsets* of H_α, so called because the destruction of all the hubs of α in $S \in \mathscr{C}_\alpha$ will disable network α. It is natural to assume, and we will, that the $\mathscr{C}_\alpha$ are *increasing* in the sense

$$S \in \mathscr{C}_\alpha \;\&\; S \subset T \implies T \in \mathscr{C}_\alpha. \tag{3}$$

Country II needs to disable *both* networks. It knows only the set $H = H_R \cup H_B$ of all sites and has no information regarding the pairs $(H_\alpha, \mathscr{C}_\alpha)$ other than that they exist. For each $h \in H$ it has weapons, r_h and b_h, which can destroy a red hub or a blue hub respectively at site h, provided the hub exists there.[1] It also possesses several "h-missiles" which are trained to fire r_h and b_h at $h \in H$; and each h-missile can carry either r_h or b_h or both, but the probability that it will hit h is p_h, independent of the outcome of other missiles. Moreover, the missiles must be fired rapidly before it can be known which ones hit their targets. The military is not concerned with the costs of the weapons or of the missiles[2]. It is pondering over its $2^{|H|}$ strategies, one for every $S \subset H$, where the S-strategy consists of firing the weapons jointly at the sites in S and separately at those in $H \setminus S$.

Which strategy should the military employ? And would it be of benefit for country II to conduct espionage to find out $(H_\alpha, \mathscr{C}_\alpha)$ in order to fine-tune its strategy based on that information?

As in the previous example, there seem to be some advantages to firing separately. For suppose the weapons are fired jointly at all h, and it turns out that the set S of targets hit is in $\mathscr{C}_R$ but not in $\mathscr{C}_B$, with the upshot that Country II fails in its objective. Perhaps its prospects might have been better had it fired separately. Conditional on the realisation of S, there would still have remained a positive probability of hitting a critical set T in $\mathscr{C}_B$.

Nevertheless, if $\Pi_2(S)$ denotes the probability that both networks get disabled under the S-strategy then we prove the following.

[1] e.g., the hubs of R (or, B) are above (or, below) ground; and r_h detonates above, while b_h burrows into the earth and detonates below.

[2] However, see Remark 7.2.

Claim 4 *The probability* $\Pi_2(S)$ *is an increasing function of S.*

Proof. See Section 4. ■

Corollary 5 *The H-strategy maximizes the probability of disabling both networks.*

Remark 6 *Note that this result holds no matter what* $(H_R, \mathscr{C}_R)$ *and* $(H_B, \mathscr{C}_B)$ *are. Thus espionage is of no benefit. If the military* were *concerned about the costs of* r_h, b_h *then of course it would be of benefit to know* H_R, H_B *in order to avoid firing* r_h, b_h *at* $H \diagdown H_R$, $H \diagdown H_B$ *respectively.*

It turns out that firing jointly increases not just the probability of disabling both networks, but *also* the probability of disabling neither. What decreases is the probability of disabling exactly one network. Here is the precise statement.

Claim 7 *Let* $F(S)$ *and* $G(S)$ *be the probabilities of disabling neither network, or exactly one network, under the S-strategy; then F is an increasing function and G is a decreasing function of S.*

Proof. See Section 4. ■

3 Example 3: Corporate merger

Each individual $h \in H$ owns shares in company $\alpha \in \{A, B\}$, and there are no shareholders outside of H. The *voting game* (aka "simple game", see [13]) in company α is described by a function

$$f_\alpha : \mathscr{H} \longrightarrow \{0, 1\}$$

where $\mathscr{H}$ denotes the collection of all coalitions (subsets) of H. The interpretation is that coalition $S \subset H$ is *winning* (resp., *losing*) *in company* α if $f_\alpha(S) = 1$ (resp., $f_\alpha(S) = 0$). We naturally assume that f_α is *increasing,* i.e., adding voters to a winning coalition cannot make it losing; and (to avoid trivialities) that the empty coalition $\emptyset$ is losing while the grand coalition H is winning[3].

[3]A canonical example is provided by a *weighted voting game* in which player h has $r^h_\alpha \geq 0$ votes in company α, equal to his shares in α (adhering to the "one dollar, one vote" principle), and $f_\alpha(S) = 1$ if, and only if, $\sum_{\alpha \in S} r^h_\alpha \geq q_\alpha$ for some "quota" $0 < q_\alpha \leq \sum_{\alpha \in H} r^h_\alpha$.

The possibility has arisen of the merger of A and B but this requires the approval of owners of both companies, i.e., merger must be voted for by a winning coalition in both A and B.

We postulate (in the spirit of [5] and [10], see also [6] for a detailed survey) that each h has an exogenously given probability p_h of returning his ballot when called upon to vote Y (yes) in favor of merger; and that these probabilities are independent across $h \in H$ and across the different occasions on which any particular h may be asked to return a ballot.

The management of both companies are strongly in favor of the merger. They can send two separate ballots to any $h \in H$, one of which is a vote of Y in company A, and the other a vote of Y in company B (and this does seem to be the norm in practice, where the decision-making inside any company is kept independent of other companies). However, an interesting alternative presents itself in the current context: they can send a "joint ballot" to h, with the explicit understanding that if h returns that ballot it will mean that h voted Y in *both* A and B.

Thus, yet again, there is an S-strategy for every $S \subset H$: send joint ballots to $h \in S$ and separate ballots to $h \in H \diagdown S$. If $\Pi_3(S)$ is the probability of merger under the S-strategy then the management is looking to maximize $\Pi_3(S)$.

Which strategy $S \subset H$ will maximize the probability of merger?

The advantages of risk-diversification notwithstanding, we show

Claim 8 *The merger probability $\Pi_3(S)$ is an increasing function of S.*

Proof. See Section 4. ■

Corollary 9 *The H-strategy maximizes the probability of a merger.*

4 Convolution of increasing functions

We now formulate and prove the main result that underlies all our examples.

Let $\mathscr{H}$ be the set of subsets of a finite set H. For S in $\mathscr{H}$ we define a probabilty measure μ_S on $\mathscr{H} \times \mathscr{H}$ as follows: $\mu_S(S_1, S_2) = 0$ if $S \cap S_1 \neq S \cap S_2$, otherwise

$$\mu_S(S_1, S_2) = P(S, S_1) P(H \setminus S, S_1) P(H \setminus S, S_2)$$

where $P(X,Y) := \prod_{h \in X \cap Y}(p_h) \prod_{h \in X \setminus Y}(1 - p_h)$. Then $\mu_S(S_1, S_2)$ is precisely the probability that the following random procedure leads to the pair (S_1, S_2):

- For each h in S we toss a coin with probability p_h of landing heads; if heads we include h in both S_1 and S_2, and if tails then we exclude h from both sets.
- For each h in $H \setminus S$ we toss the coin once to decide whether to include h in S_1, and once again, independently, to decide whether to include h in S_2.

Definition 10 *If f and g are real valued functions on $\mathscr{H}$ then we define*

$$f \star g(S) := \textstyle\sum_{S_1,S_2 \subseteq H} f(S_1)g(S_2)\mu_S(S_1,S_2). \tag{4}$$

As before, we say that f is *increasing* if $S \supseteq T$ implies $f(S) \geq f(T)$.

Theorem 11 *If f and g are increasing functions then so is $f \star g$.*

Proof. It suffices to show $f \star g(S) \geq f \star g(T)$ in the case where $S \setminus T$ consists of a single element, h, say. Then the two coin tossing procedures agree on $H' = H \setminus \{h\}$ and by grouping terms we can rewrite

$$\begin{aligned} f \star g(S) &= \textstyle\sum_{T_1,T_2 \subseteq H'} A(T_1,T_2)\,\mu'_T(T_1,T_2) \\ f \star g(T) &= \textstyle\sum_{T_1,T_2 \subseteq H'} B(T_1,T_2)\,\mu'_T(T_1,T_2) \end{aligned}$$

where μ'_T is the measure on H' induced by T, and $A = A(T_1,T_2)$ and $B = B(T_1,T_2)$ are given explicitly by the formulas

$$\begin{gathered} A = (1-p)\,ab + pa'b', \quad B = \left[(1-p)a + pa'\right]\left[(1-p)b + pb'\right] \\ \text{with } a = f(T_1), a' = f(T_1 \cup \{h\}), b = g(T_2), b' = g(T_2 \cup \{h\}), p = p_h. \end{gathered}$$

We claim that in fact $A(T_1,T_2) \geq B(T_1,T_2)$ for all T_1,T_2. Indeed, an easy calculation shows that

$$A - B = p(1-p)\left(a'-a\right)\left(b'-b\right).$$

Now since f and g are increasing we have $a' \geq a, b' \geq b$, and hence $A - B \geq 0$. ■

For the two extreme cases $S = H$ and $S = \emptyset$ we have

$$f \star g(H) = \mathrm{Exp}(fg), \quad f \star g(\emptyset) = \mathrm{Exp}(f)\,\mathrm{Exp}(g),$$

where Exp is the expectation with respect to the measure μ on $\mathscr{H}$ defined by

$$\mu(S) = \textstyle\prod_{h \in S} p_h \prod_{h \notin S}(1-p_h) = P(H,S). \tag{5}$$

Thus we obtain the following well-known inequality due to Harris [8].

Corollary 12 *If f and g are increasing then we have*

$$\operatorname{Exp}(fg) \geq \operatorname{Exp}(f)\operatorname{Exp}(g). \tag{6}$$

If $\mathscr{F} = \{f_i, i \in I\}$ is an *set* of functions and $\pi : I_1 \sqcup \cdots \sqcup I_k$ is a partition of the index set I then we define $E_{\mathscr{F}}(\pi) = \operatorname{Exp}\left(\prod_{i \in I_1} f_i\right) \cdots \operatorname{Exp}\left(\prod_{i \in I_k} f_i\right).$

Corollary 13 *Let $\mathscr{F}$ be a finite set of non-negative increasing functions and let π and π' be partitions of I such that π' refines π then we have $E_{\mathscr{F}}(\pi') \leq E_{\mathscr{F}}(\pi)$.*

Proof. If $f_1, \ldots, f_k$ are nonnegative and increasing, then so is their product. Now the result follows by iterated application of (6). ■

We now use these results to prove the various claims in the previous examples.

4.1 Proof of Claim 1

Proof. It is easy to see that the output for the S-strategy is $\Pi_1 = F_1 \star F_2$. Thus the result follows from Theorem 11. ■

Remark 14 *The proof shows that Claim 1 is* equivalent *to Theorem 11.*

4.2 Proof of Claim 4

Proof. Let f and g be the characteristic functions of $\mathscr{C}_R$ and $\mathscr{C}_B$, then by assumption f, g are increasing. We have $f(S_1)g(S_2) = 1$ or 0 according as the pair (S_1, S_2) does or does not succeed in disrupting both networks. It follows that the success probability for the S-strategy is $\Pi_2 = f \star g$, which is increasing by Theorem 11. ■

4.3 Proof of Claim 7

Proof. The probabilty $F(S)$ of disabling neither network is given by

$$F = (1-f) \star (1-g) = (f-1) \star (g-1),$$

where f and g are as in the previous proof. Since $(f-1)$ and $(g-1)$ are increasing functions so is F. This implies that $G = 1 - \Pi_2 - F$ is a decreasing function. ■

4.4 Proof of Claim 8

Proof. It is easy to see that the success probability of the S-strategy is $\Pi_3 = f_A \star f_B$, where f_A and f_B are the voting functions of companies A and B. Now f_A and f_B are increasing by assumption and thus by Theorem 11 so is Π_2. ■

5 Example 4: Stochastic production - many inputs

We now consider production with inputs from a finite commodity set K.

An entrepreneur sources her inputs from a set H of suppliers. Supplier h agrees to supply commodities from a subset K^h of K — the sets K^h need not be disjoint. She further enters into a "shipping contract" P^h with h, which is a partition of K^h into a disjoint union of shipments $K^h = K_1^h \sqcup \cdots \sqcup K_l^h$. However there is uncertainty in shipping and shipment K_i^h arrives with independent probability p_h.

After receiving the shipments, she produces an output given by

$$F(\mathbf{S}) = \prod_{k \in K} F_k(S_k),$$

where S_k is the set of players whose shipment of k arrives successfully, $\mathbf{S}$ is the "success" tuple $(S_k)_{k \in K}$, and the F_k are non-negative increasing functions on subsets of H, that is to say $F_k(S) \geq F_k(S')$ if $S \supset S'$. Her goal is to choose the contract tuple $\mathbf{P} = \left(P^h\right)_{h \in H}$ which maximizes the expected output $\Phi(\mathbf{P})$.

We now formulate a generalization of Claim 1, which involves a partial order on shipping contract tuples. If $\mathbf{P} = \left(P^h\right)_{h \in H}$ and $\mathbf{Q} = \left(Q^h\right)_{h \in H}$ are two such tuples we write $\mathbf{P} \succeq \mathbf{Q}$ if P^h is *coarser* than Q^h for all h, that is if each P^h-shipment is a union of Q^h-shipments.

Claim 15 *The expected output is monotonic:* $\mathbf{P} \succeq \mathbf{Q}$ *implies* $\Phi(\mathbf{P}) \geq \Phi(\mathbf{Q})$.

Proof. See section 6.1. ■

Remark 16 *For two commodities Claim 15 reduces to Claim 1.*

The maximal element for $\succeq$ is the "coarse" tuple $\mathbf{C} = \left(C^h\right)_{h \in H}$ where C^h is a single shipment of all commodities in K_h.

Corollary 17 *The coarse tuple* $\mathbf{C}$ *maximizes expected output.*

Remark 18 *Suppose supplier h agrees to send x_k^h units of commodity k. Then the total amount of k received by the entrepreneur is $x_k = \sum_{h \in S_k} x_k^h$. If we set $F_k(S_k) = (x_k)^{\alpha_k}$ for $a_k > 0$ then the F_k are increasing functions, and $F(\mathbf{S}) = \prod_{k \in K} (x_k)^{\alpha_k}$ is the Cobb-Douglas production function.*

6 A game-theoretic generalization

We now recast the previous example as a strategic game among the suppliers. We use the same setup, $K, H, K^h, P^h, p^h, S_k, \ldots$, with two differences.

1. We think of the partition P^h of K^h as a *strategic* choice of shipments by h.

2. If the success tuple is $\mathbf{S} = (S_k)_{k \in K}$ then player h receives the payoff

$$F^h(\mathbf{S}) = \prod\nolimits_{k \in K} F_k^h(S_k),$$

where the F_k^h are non-negative increasing functions that may depend on h.

We will show that in this game[4] the coarse strategy $\mathbf{C}$ is *dominant* in a very strong sense. For this, we consider the following scenario. Suppose the shipping tuple $\mathbf{P}$ has been played, in which P^h corresponds to the partition $K^h = K_1^h \sqcup \cdots \sqcup K_l^h$ with $l \geq 2$. Player h is informed of the success/failure of all shipments, including his own, except for K_1^h and K_2^h. Let π be his expected payoff conditional on this information, and let π' be his expected payoff if he chooses to combine the commodities in K_1^h and K_2^h into a single shipment before sending them off.

Proposition 19 *In the above scenario we have $\pi' \geq \pi$.*

Proof. Let K' be the set of commodities not in $K_1^h \cup K_2^h$, then the sets $S_k, k \in K'$ are constant under the conditioning assumption, and so is the product $c = \prod_{k \in K'} F_k^h(S_k)$. The products $a = \prod_{k \in K_1^h} F_k^h(S_k)$ and $b = \prod_{k \in K_2^h} F_k^h(S_k)$ have two possible values $a_1 \geq a_0$ and $b_1 \geq b_0$ corresponding to the arrival/non-arrival of shipments K_1^h and K_2^h, respectively. When sent separately these arrive with (independent) probability $p = p_h$, while sent together they arrive together with probability p. The payoff is the expectation of the product abc, and so we get

$$\pi' = [(1-p)\,a_0 b_0 + p a_1 b_1]\,c, \quad \pi = [(1-p)a_0 + pa_1]\,[(1-p)b_0 + pb_1]\,c.$$

Now a straightforward algebraic calculation shows that

$$\pi' - \pi = p\,(1-p)\,(a_1 - a_0)\,(b_1 - b_0)\,c$$

Every factor in this expression is non-negative, which implies $\pi' - \pi \geq 0$. ■

[4] See Remarks 7.3, 7.4, 7.5 for more details on the game.

Theorem 20 *Fix a choice of strategies by all players except h and let $\pi(P)$ denote the expected payoff to h if he chooses the partition P. If P is coarser than Q then we have $\pi(P) \geq \pi(Q)$. Thus the coarse partition is a dominant strategy for h.*

Proof. We can go from any partition to a coarser partition in a sequence of steps, where at each step we combine two parts of a partition into a single part. Clearly it is enough to show that the desired inequality holds at each step of this procedure. But this follows from the previous proposition. ■

6.1 Proof of Claim 15

Proof. We specialize the strategic model to the "symmetric" case in which $F_k^h = F_k$ for all h. Then each player's payoff is the same as that of the principal in stochastic production model of the previous section. Thus the two optimization problems are the same, and hence Claim 15 follows from Theorem 20. ■

6.2 Another proof of Theorem 11

Proof. Claim 15 implies Claim 1, which is equivalent to Theorem 11. ■

7 Concluding Remarks

7.1 Robust Optimality

Pooling is optimal *for all possible characteristics* of the population under consideration, e.g., $(x_h, y_h, p_h)_{h\in H}$ in Example 1, $(H_h, \mathscr{C}_h)_{h\in\{A,B\}}$ in Example 2, and so on. In short, the H-strategy is not only optimal but *robustly optimal*. And therein lies its full value: pooling can be invoked with impunity, without having detailed information of the population characteristics.

7.2 Costs

We have assumed various monetary costs (such as those of firing missiles in Example 1, or of shipping inputs in Example 2, etc.) to be zero. Suppose these costs existed, but with "economies of scale", i.e., the cost of joint action is less than the sum of the costs of the separate actions. Then the pooling strategy is even more efficient, as it diminishes costs, *over and above* the favorable probabilistic implications of Theorem 11.

7.3 Nash Equilibrium and Dominant Strategies

For completeness' sake, we review some standard game-theoretic definitions. First recall that once each supplier h chooses his strategy P^h (i.e., a partition of K^h), a probability distribution is induced on success tuples $\mathbf{S} = (S_k)_{k\in K} \in \mathscr{H}^K$ via the independent arrivals of the elements of the partitions in $\mathbf{P} = \left(P^h\right)_{h\in H}$; and that the payoff to h is the expectation $\Phi^h(\mathbf{P})$ of $\prod_{k\in K} F_k^h(S_k)$ w.r.t. this probability distribution. We are now ready for the definitions (and the subsequent remarks below). A strategy P^h of player h is a *best reply* to the strategy-selection $\left(P^i\right)_{i\in H\diagdown\{h\}}$ of the other players, if P^h maximizes the payoff to h over all his strategies, conditional on the fixed $\left(P^i\right)_{i\in H\diagdown\{h\}}$. The H-tuple of strategies $\mathbf{P} = \left(P^h\right)_{h\in H}$ is a *Nash Equilibrium* (NE) if P^h is a best reply to $\left(P^i\right)_{i\in H\diagdown\{h\}}$ for every $h \in H$. Next, P^h is a *dominant strategy* of h if P^h is a best reply to *every* strategy-selection of the players in $H\diagdown\{h\}$. Finally P^h is a *strictly dominant strategy* of h if P^h is the *unique* best reply to every strategy-selection by the players in $H\diagdown\{h\}$. It is obvious that if P^h is a strictly dominant strategy for every $h \in H$, then $\mathbf{P} = \left(P^h\right)_{h\in H}$ is the unique NE of the game. It is also obvious that if each F_k^h is assumed to be *strictly* increasing, then the coarse partition is *strictly* dominant for every player (This follows from the fact that $\pi' - \pi$ is strictly positive in the proof of Proposition 19). In this case the coarse partitions constitute the *unique* Nash Equilibrium of the game.

7.4 *Ex Post* Optimality

Our proof of Theorem 20 shows that the coarse partition not only maximizes the expectation of $\prod_{k\in K} F_k^h(S_k)$ for player h, but in fact maximizes his expected payoff conditional on *every realization* of the success tuples in $\widetilde{\mathscr{H}}^K$ due to the other players (where $\widetilde{\mathscr{H}}$ denotes the collection of subsets of $H\diagdown\{h\}$). In other words, the coarse strategy is not just dominant in the standard sense (*ex ante* optimal), but in a much stronger sense (*ex post* optimal).

7.5 Incentive Compatibility

Theorem 20 is not only useful as a generalization of Theorem 11 (with a transparent proof), but it considerably bolsters the economic plausibility of our examples. To this end, consider Example 4. It is not easy for the entrepreneur to *implement* the optimal contract with her agents. She would need to monitor their behavior and institute costly punishment for anyone who breaks the contract, i.e., takes it into his head to choose a partition other than the coarsest. Moreover she would need to make it credible to her agents that the punishment will be forthcoming so that it acts as a deterrent. All this is not achieveable without considerable cost and effort, if it is achievable at all. However, our game-theoretic approach provides a way out of this impasse. The entrepreneur can simply announce that she will part with a (tiny!) fraction κ^h of her profit $\prod_{k\in K} F_k(S_k)$ to each supplier h. This engenders a game among the suppliers with payoffs $F^h(\mathbf{S}) = \kappa^h \prod_{k\in K} F_k(S_k)$ to $h \in H$, i.e., a game with common payoffs (upto scalar multiplication). In this game it is a dominant strategy for each h to choose his coarsest partition by Theorem 20. Thus the optimal contract of the principal is implemented by her agents of their own accord, at virtually no cost to her!

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