Proportional Systems with Free Entry. A Citizen-Candidate Model

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Abstract

We analyze the equilibrium of a proportional electoral system with free entry in a citizen candidate model. In proportional systems the policy outcomes are typically decided through legislative bargaining and a perspective entrant has to worry about the governing coalitions that will be able to reach 50% of the seats. We show that there are equilibria with medium-sized parties, i.e. no party has absolute majority but the number of parties is relatively small. However, when the number of seats is sufficiently large, all equilibria must have at least 4 parties. We also discuss the impact of variations of the electoral formula, such as the introduction of of thresholds.

1 Introduction

According to the Duverger's Hypothesis, proportional systems tend to have a larger number of parties than majoritarian ones. But how many parties can we expect to see in a proportional system and why? Many models of the proportional system assume a fixed number of parties (see for example Baron [5], Cho [9], Cho [10] and Indridason [20]). While models with a fixed number of parties are very useful for a number of questions, they are not useful when the question is '*how many parties can*

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we expect in a proportional system?'. Furthermore, the assumption of free entry and exit is the natural one when we want to evaluate the consequences of modifications of the electoral formula, e.g. the imposition of a minimum legal threshold of votes for representation.

When there is free entry the standard Downsian model of office-seeking politicians does not offer interesting answers. Even ignoring potential equilibrium existence problems, a model in which the only goal of parties and politicians is to win seats leads to the prediction that there will be as many parties as seats available[1]. Thus, with free entry and purely office–seeking politicians we should observe many very small parties and no medium-sized or large parties. This is at odds with the empirical evidence. Diermeier and Merlo [17] report that in 313 post-war elections in 11 democratic countries the average and median number of represented parties was, respectively, 7.35 and 7. This is typically far below the maximum number of parties that can obtain representation. Many proportional systems have parties of significant size and in general the number of relevant parties is not very large. A model of the proportional system should therefore be able to produce equilibria in which at least some of the parties are either medium-sized or large, although no party gets more than 50% of the seats. Alternatives to the pure Downsian model must be explored.

The citizen-candidate model offers such an alternative. This paper considers a polity with a continuum of citizens with ideal points distributed on a compact interval. Decisions are taken by majority in a parliament with a finite number M of seats. Each citizen can form a party and run for election. Seats are distributed proportionally. The platform of the party is identified by the ideal point of the party founder. Citizens are interested in the policy that will be implemented and have private costs and benefits for forming a party and obtaining representation. Thus, the two main assumptions of the paper are that voting is sincere but the decision to form a party is strategic.

An important ingredient of any model of the proportional system is the legislative bargaining game that determines the policy. In many elections no party obtains an absolute majority of seats, so strategic decisions about entry have to be taken looking at the effect on the composition of the Parliament and the implied policy outcomes. We consider a 'random formateur' model à la Diermeier and Merlo [17]. If a party has a share of seats greater than 50% then that party is chosen with probability 1,

[1] This statement should be qualified. We consider a model in which each party proposes a single political platform and all the candidates belonging to a party commit to implement that platform. In principle different candidates inside a party may differentiate their policy platforms, a strategy particularly attractive in 'open list' systems. We consider a 'closed list' system in which the party leader has control over the seats won by the party.

otherwise each party is selected with a probability equal to the share of the seats of the party. Once selected, the formateur has to form a coalition achieving an absolute majority of the seats. Once the coalition is formed, the policy preferred by each party belonging to the coalition is chosen with probability equal to the share of seats of the party over the number of seats *of the coalition*.

One important question is whether equilibria with large or medium-sized parties exist. We show that this is in fact the case. The intuition is similar to the one discussed in Indridason [20]: Citizens voting a large but not dominant party are unwilling to break away and form a different party because, under the new political configuration, they may be excluded more often from the governing coalitions[2]. On average a breakaway party will improve utility in terms of policy when it is included in a governing coalition, but this is countered by a lower probability of belonging to the governing coalition as well as by a worsening of the policies implemented when excluded. If this effect is strong enough, it will more than balance any gain in utility obtained by winning seats.

The rest of the paper is organized as follows. The model is described in section 2, which also contains some preliminary results. In particular, we show that when the number of seats is sufficiently large any equilibrium must have at least 4 parties. Typically, the model has multiple equilibria. Section 3 discusses the class of equilibria that we consider most interesting and more frequently observed, i.e. equilibria in which no party has a majority of seats but at least some parties are far larger than the minimum needed for representation. We then consider in section 4 the effect of the introduction of thresholds for representation, i.e. rules that deny representation to parties receiving less than a minimum share of the votes. Section 5 compares our results to the ones obtained in the formal literature on proportional systems, focusing on the role played by the different assumptions made. The section also discusses some possible extensions of the theory left for future work. Section 6 contains the conclusions. Appendix I contains the proofs and Appendix II contains the computations of two examples appearing in the paper.

2 The Model

A continuum of voters, with ideal points distributed on the interval $[0,1]$, elect a finite number M of representatives. There is a single electoral college and the electoral system is proportional. More specifically, we consider a version of the Largest

[2]Indridason [20] does not allow for free entry and keeps the number and locations of parties fixed. He assumes strategic voting.

Remainder rule. Let m be the number of parties contesting the election and V_i the number of valid votes received by party i. Let $V = \sum_{i=1}^{m} V_i$ and define $v_i \equiv \frac{V_i}{V}$ the share of the vote of party i. Seats are assigned according to the following procedure.

- **Stage** 0**.** Each party i is assigned $S_0^i = \lfloor v_i M \rfloor$ seats, where $\lfloor x \rfloor$ is the integer part of x. If $\sum_{i=1}^{m} S_0^i = M$ then stop. Otherwise, compute the rest $R_0^i = v_i M - S_0^i$ for each party and go to the next stage.

- **Stage** k**.** If i is such that $R_{k-1}^i = \max_{j=1,\ldots m} R_{k-1}^j$ then $S_k^i = S_{k-1}^i + 1$ (if there are multiple parties with the highest rest R_{k-1}^i then one of them is chosen randomly with equal probability). Otherwise, $S_k^i = S_{k-1}^i$. If $\sum_{i=}^{m} S_k^i = M$ then stop. Otherwise, compute the rest $R_k^i = v_i M - S_k^i$ for each party and go to the next stage.

Since M is finite the procedure converges in a finite number of rounds. The procedure is equivalent to determining the quota $Q = \frac{V}{M}$ needed for a seat, then assigning seats to each party i according to the integer part of $\frac{V_i}{Q}$ and finally distributing the remaining seats according to the highest rests.

Citizens' ideal points are distributed over $[0, 1]$ according to the cumulative distribution $F(x)$, with a continuous density $f(x)$. Each citizen can form a party. We follow Osborne and Slivinski [22] and assume that it is impossible to commit to a policy, so everybody expects that a candidate with ideal point x will implement policy x. Citizens prefer policies closer to their ideal point. Furthermore, they pay a cost $c \geq 0$ whenever they decide to form a party. If they form a party and they obtain seats then they obtain *parliamentary rents* which are proportional to the share of seats they obtain in an election.

The strategy set of each citizen is simply $\Sigma = \{0, 1\}$, where $\sigma = 0$ means that the citizen is not running and $\sigma = 1$ means that the citizen is forming a party and participating to the election. The utility function of a citizen located at x when a policy y is implemented and a share s_x of seats is obtained in case of participation to the election is given by

$$u_x(s_x, y, \sigma) = (\alpha s_x - c)\,\sigma - |x - y|\,.$$

If the election has zero candidates then we assume a default utility $\underline{u}_x$. The parameter α reflects the relative weight that citizens assign to parliamentary rents. The constraints on the value of the parameters are collected in the next assumption.

Assumption 1 *$\alpha \geq 0$, $0 \leq c < 1$ and $\underline{u}_x < -1$ for each x.*

The assumption implies that in equilibrium there is always at least one party running. As α goes to zero citizens are only motivated by policy and parliamentary rents become irrelevant, while as α goes to $+\infty$, the parties care almost exclusively about parliamentary rents.

2.1 The Political Game

In a proportional system, unless a party gets an absolute majority of the seats, the policy will be determined by a compromise among parties forming a governing coalition. Thus, in this case, it becomes crucial to model the legislative bargaining game that takes place after the election. We adopt the 'random formateur' model proposed by Baron and Diermeier [6], which works as follows (possible alternatives are discussed in section 5). Whenever a party gets the absolute majority of seats then that party forms the government. When no party has the majority of seats then each party has a probability of becoming the formateur equal to its share of the seats. More precisely, a party becomes formateur with probability 1 if the number of seats S_i assigned to that party is such that $S_i > \lfloor M/2 \rfloor$. Otherwise, the party becomes the formateur with probability $s_i = S_i/M$. Overall, the stage game we consider is the following.

Party formation stage Each citizen decides whether or not to form a party and run. Each party is characterized by a political platform, given by the preferred point of the citizen founding the party.

Electoral stage Let $\mathcal{N}$ be the set of parties entering at stage 1. If $\mathcal{N} = \emptyset$ (no party runs) then the polity breaks down, each citizen gets utility $\underline{u}_x$ and the game ends. Otherwise, each citizen votes sincerely for the closest party. Let $v = \{v_i\}_{i \in \mathcal{N}}$ be the vector of vote shares, i.e. v_i is the share of the popular vote of party i. Let $\mathcal{J} = \{1, \ldots, J\}$ be the finite set of parties obtaining parliamentary representation given the vector of vote shares $v = \{v_i\}_{i \in \mathcal{N}}$ ($\mathcal{J}$ is a, possibly proper, subset of $\mathcal{N}$). Call s_j the share of parliamentary seats of party j and $s = \{s_j\}_{j \in \mathcal{J}}$ the vector of seats. Define $s^{\max} = \max_{j \in \mathcal{J}} s_j$ the seat share of the most voted party.

Legislative bargaining stage Party i is chosen as formateur with probability $p_i(s)$ defined as

$$p_i(s) = \begin{cases} 0 & \text{if} \quad s^{\max} > 0.5 \quad \text{and} \quad s_i \neq s^{\max} \\ s_i & \text{if} \quad s^{\max} \leq 0.5 \\ 1 & \text{if} \quad s^{\max} > 0.5 \quad \text{and} \quad s_i = s^{\max} \end{cases} \tag{1}$$

Once a formateur j is chosen, the formateur chooses a subset $\mathcal{H} \subset \mathcal{J}$ of parties and proposes the formation of a coalition government. The subset $\mathcal{H}$ has to be such that $\sum_{j \in \mathcal{H}} s_j > 0.5$ (we ignore the possibility of minority governments, although that possibility could be introduced at the cost of complicating the model). The chosen parties sequentially[3] say 'accept' or 'no'. If all the parties in $\mathcal{H}$ accept then a government is formed, otherwise all citizens receive the default utility $\underline{u}_x$.

Policy implementation stage The policy implemented is a probability distribution over the ideal points of the parties in $\mathcal{H}$, defined as follows. Let $s_{\mathcal{H}} = \sum_{j \in \mathcal{H}} s_j$ be the size of the parliamentary majority for coalition $\mathcal{H}$. The probability of selecting the ideal point x_i of party $i \in \mathcal{H}$ is $q_i^{\mathcal{H}} = s_i / s_{\mathcal{H}}$. Notice that when $s^i > 0.5$ the resulting coalition is $\mathcal{H} = \{i\}$ (one-party government) and the policy is x_i with probability 1.

The outcome of the simple legislative game is that the formateur makes a proposal to the coalition generating the highest expected utility and the members of the coalition say yes. The assumption that the policy is determined according to a probability distribution over policy positions is equivalent to assuming that the parties cannot bargain *ex ante* over the policy to be implemented. We are essentially assuming that commitment is not possible, so that the policy is influenced *ex post* by the composition of the coalition and the bargaining power is well approximated by the importance of a party for the governing coalition.

The assumption that a party with the absolute majority of seats is put in charge of government with probability 1 is natural. The rule that we adopt when no party has the majority of seats is not natural and it should be seen as a stylized representation of the decision process. In many parliamentary regimes, when there is no party with a clear majority the formateur is chosen by some higher authority, e.g. the President of the Republic, usually considering the probability that a party may be able to form a viable coalition. Other rules can be used (for example Indridason [20] assumes that the formateur is the party with the most votes) and we will discuss the relevance of our assumption in section 5.

2.2 The Equilibrium Concept

We will consider subgame perfect equilibria of this game. Actually, the only strategically relevant part is the entry (party formation) stage. Once the set of entrants

[3] We assume that parties accept sequentially in order to avoid irrelevant multiple equilibria that may occur when the parties act simultaneously.

has been determined, the probability distribution on outcomes follows automatically, since there is only one subgame perfect equilibrium for each selection of the formateur. In the classic citizen candidate model, a citizen must enter at its ideal point. This may lead to knife-edge equilibria in which the position of the party causes ties in the number of seats and a small movement in the position would lead to strong benefits in terms of expected utility.

We want to avoid these situation, so we will add a robustness requirement similar to the one used in Brusco and Roy [7]. Formally, if a candidate has ideal point x we assume that the party can choose the platform in the interval $(x-\delta, x+\delta)$, and study equilibria that survive as δ goes to zero. This leads us to introduce the following equilibrium notion. For a given set of parties $\mathcal{N}$ let $U^x(\mathcal{N})$ the expected utility of a citizen located at x when the parties in $\mathcal{N}$ participate to the election.

Definition 1 *A robust equilibrium of the electoral game is a set of entrants $\mathcal{N}$ that satisfies the following properties:*

1. ***No Entry**. For each $x \notin \mathcal{N}$, $U^x(\mathcal{N}) \geq U^x(\mathcal{N} \cup x)$.*
2. ***No Exit.** For each $x \in \mathcal{N}$, $U^x(\mathcal{N}) \geq U^x(\mathcal{N} \setminus x)$.*
3. ***Robustness.** For each $x \in \mathcal{N}$, there exists $\delta > 0$ such that $U^x(\mathcal{N}) \geq U^x((\mathcal{N} \setminus x) \cup x')$ for each $x' \in (x-\delta, x+\delta)$.*

In words, a political equilibrium is a situation in which no citizen wants to create a new party and each existing party prefers to maintain the current platform rather than either disbanding or slightly changing its position. More precisely, the robustness requirement says the following. Consider a party located at x entering the electoral competition, so that $x \in \mathcal{N}$. Suppose that, instead of entering with platform x, the party can commit to platform x'. By withdrawing as x and entering as x' the set of entering party is $(\mathcal{N} \setminus x) \cup x'$. Our robustness requirement says that an entering party does not want to do that as long as x' is sufficiently close to x, i.e. each citizen prefers to enter with the preferred platform rather than with a platform which is very close to it but different. This requirement eliminates knife-edge equilibria in which, for example, the party located at x may be excluded or included in certain coalitions just because the other parties are indifferent between including or excluding it. It essentially guarantees that all our equilibria are in pure strategies.

2.3 General Results

We will look at robust equilibria of the electoral game. We start with a somewhat obvious observation regarding the cost of running c. Let $\widehat{S}_M$ be the highest number of seats such that $\widehat{S}_M \leq \frac{M}{2}$. Thus, $\widehat{S}_M + 1$ is the minimum number of seats needed to get strictly more that 50% of the seats. Define $\widehat{s}_M = \widehat{S}_M / M$.

Proposition 1 *There is no equilibrium in which no party runs. In a parliament with M seats, if the cost of entry is $c < \alpha \widehat{s}_M$ then there are no single party equilibria.*

The proposition states that, unless the entry cost is prohibitive, there must be multiple parties entering. When there is a single party then an entrant can win a large share of the seats, and with a low c this is enough to make entry profitable. This should also be seen as the most realistic case, since $c < \alpha \widehat{s}_M$ says that the cost of running is no more than the benefit of obtaining almost the absolute majority of seats.

Since the consequences of having a high c are somewhat obvious, we will dedicate most of the paper to discuss the case of a low c. In fact, we will assume $c = 0$ for the rest of paper and discuss the effect of a strictly positive c in Section 5.

Remark. We assume that the entry cost is the same for each citizen. This is obviously unrealistic and we can expect a lot of heterogeneity. However, all the results of the paper survive without any change if we assume that there is a dense subset $A \subset [0, 1]$ of zero measure, e.g. the set of rational numbers in $[0, 1]$, of *politically active* citizens with a very low cost c, while the rest of the population is made by politically passive citizens with a high c. This interpretation also makes it more natural to assume that voting is sincere while the entry decision is strategic. Politically active citizens, a small fraction of the population, are the only ones making strategic decisions. Other citizens do not pay much attention to the political process and simply vote for the closest party.

Our first result provides a characterization of equilibria in which a single party is dominant, i.e. it obtains at least $\lfloor \frac{M}{2} \rfloor + 1$ seats (absolute majority). Situations in which a dominant party emerges in a proportional system are not frequent, but some cases have been historically observed[4].

[4]Examples include the Social Democrats in Sweden in 1968, the CDU-CSU (taken as a single party) in Germany in 1957, the People's Party in Spain in 1996 and the Socialist Party in Spain in 1982 and 1986.

Proposition 2 *Suppose $c = 0$. Let $\{x_1, \ldots, x_J\}$ be an equilibrium configuration such that the party located at x_k is dominant. Then the dominant party wins at most $\left\lfloor \frac{M}{2} \right\rfloor + 2$ seats and all parties with index strictly greater than $k+1$ or strictly smaller than $k-1$ have at most two seats. If a party contiguous to the dominant one (i.e. a party located at x_{k-1} or x_{k+1}) has strictly more than two seats then the dominant party has $\left\lfloor \frac{M}{2} \right\rfloor + 1$ seats.*

The intuition for the result is the following. Suppose that there is an equilibrium in which parties positioned at $\{x_1, \ldots, x_J\}$ enter and a dominant party gets a number of seats strictly larger than $\left\lfloor \frac{M}{2} \right\rfloor + 2$. Consider a citizen voting the dominant party, but with a different ideal point. In equilibrium this citizen must prefer to stay out rather than forming a new party. When $c = 0$ the main reason to stay out is the discontinuity in the policy function. If entry deprives the dominant party of the majority of seats then the policy selected becomes stochastic and this makes the entrant worse off. However, if the dominant party has a large majority of seats then there will be potential candidates at points where entry takes away some seats from the dominant party but not enough to deprive it of the majority. Since entry does not change the policy and gives to the entering party some seats, the original configuration cannot be an equilibrium. Thus, in equilibrium it must be the case that no entry of this type is profitable and this can only occur when the dominant party has a bare majority of seats.

Equilibria with a dominant party exist when the private benefits for holding seats are low, so that policy considerations are paramount when deciding whether to run or not. Citizens voting for the dominant party face a trade off: by creating their own party they may get a sizable representation in the Parliament, but this deteriorates the probability distribution on policies, since it is no longer true that the policy preferred by the dominant party is implemented with probability one.

The next proposition puts a bound on the minimum number of parties when $c = 0$ and the number of seats is large.

Proposition 3 *Suppose $c = 0$. When M is sufficiently large, all robust equilibria have at least 4 parties.*

The intuition for the result goes as follows. In a dominant party equilibrium, if there are only three parties then entry just to the right or just to the left of one of the non-dominant parties is profitable. It does not change the policy (the number of seats of the dominant party is unchanged) and it provides representation. If there are three parties and none is dominant then any coalition of two parties must have a majority.

Say that the centrist party x_2 prefers to form a coalition with x_1. Then entry just to the left of x_3 is profitable. The coalitions formed cannot change adversely, the policy implemented are essentially the same and the entrant obtains representation.

Three-party models have some popularity in the literature on proportional systems (see e.g. Austen-Smith [1], Baron [5] and Morelli [21]) as they are the easiest ones to deal with when no single party has a majority. Proposition 3 should be seen as a warning that, when the number of parties is endogenous, the results in such models should be treated with some caution.

Remark. Proposition 3 does not hold if we consider non-robust (mixed strategy) equilibria. Consider the following example. Public opinion is distributed on the interval $[0, 1]$ according to the following density function:

$$g(x) = \begin{cases} \frac{4}{3} & \text{for} \quad 0 \le x \le \frac{1}{4} \\ 0 & \text{for} \quad \frac{1}{4} < x < \frac{1}{2} - \varepsilon \\ \frac{1}{6\varepsilon} & \text{for} \quad \frac{1}{2} - \varepsilon \le x \le \frac{1}{2} + \varepsilon \\ 0 & \text{for} \quad \frac{1}{2} + \varepsilon < x < \frac{3}{4} \\ \frac{4}{3} & \text{for} \quad \frac{3}{4} \le x \le 1 \end{cases}$$

with $\varepsilon < \frac{1}{50}$. The three parties are located at $L = \frac{1}{4}$, $C = \frac{1}{2}$ and $R = \frac{3}{4}$ respectively. Given the distribution of public opinion, each party gets exactly $\frac{1}{3}$ of the vote. In Appendix II we show that this is an equilibrium, i.e. no party wants to exit and no citizen wants to form a new party. In equilibrium, when party L becomes formateur it forms the coalition $\{L, C\}$, while party R forms the coalition $\{C, R\}$. The centrist party C is indifferent between L and R and it chooses the two coalitions $\{L, C\}$ and $\{C, R\}$ with probability 50% each. This equilibrium is not robust since a slight movement to the right of L or a slight movement to the left of R would break the indifference of C and improve the expected policy outcome.

2.4 Dominant Party Equilibrium. An Example

Suppose $M = 100$ and the distribution of public opinion is uniform over th eintervl $[0, 1]$. We show that there is an equilibrium with 50 parties, one of them dominant. The dominant party is located at $x_1 = 0.505$. The remaining parties are located at $x_i = x_1 + \frac{i-1}{100}$, so that the most right-wing party is located at $x_{50} = 0.995$. The vote share of party 1 is $v_1 = \frac{0.505+0.515}{2} = 0.51$ and all the remaining parties get exactly 1% of the vote. There are no rests, so x_1 gets 51 seats and all other parties receive 1 seat each.

In this equilibrium the policy x_1 is implemented with probability 1. It is clear that no party is better off exiting, so we have to check that no entry is profitable. No entry to the right of x_2 is profitable, since the entrant would get a vote share equal to 0.5% and no seats, since at least one contiguous party would get a rest strictly higher than 0.5%. Consider entry in the interval (x_1, x_2), say at position y. The entrant obtains a share of the vote $v_y = 0.5\%$. Party x_1 obtains a vote share $v_1 = \frac{0.505+y}{2}$. Party x_2 obtains a vote share $v_2 = \frac{x_3-y}{2}$. All other parties keep getting 1%. Thus, all parties from x_3 on get exactly one seat and zero rest. Party x_1 gets 50 seats and a rest equal to

$$R_1 = \frac{0.505 + y}{2} - 0.5.$$

Party x_2 obtains a vote share $\frac{0.525-y}{2}$. This is less than 1%, so the party is assigned 0 seats and a rest equal to its vote share:

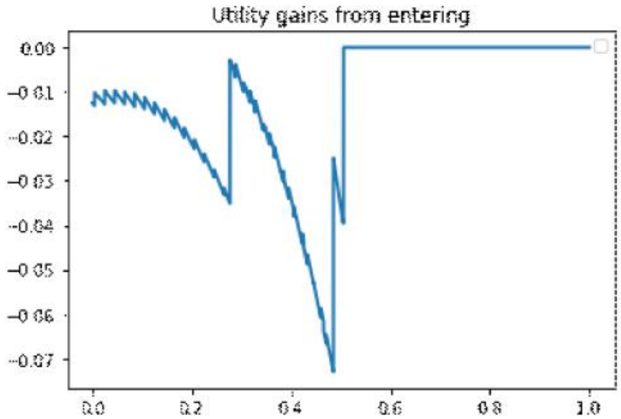
$$R_2 = \frac{0.525 - y}{2}.$$

So, after the first round, 98 seats are assigned and the only parties with strictly positive rests are x_1, y and x_2. However it is immediate to check that, since $y \in (0.505, 0.515)$, the inequality

$$\min \{R_1, R_2\} > 0.005$$

holds. This means that the entrant gets no seats, and in fact entry has no consequences.

At last, consider entry to the left of x_1. No entry can succeed in getting 51 seats. Entering at $0.505 - 2\delta$, no matter how small δ is, leaves party x_1 with a rest of $0.005 + \delta$, while the rest of the entrant is $0.005 - \delta$. Thus x_1 gets at least one seat and there must be a change in the expected outcome: instead of getting x_1 with probability 1, the entrant gets a lottery. We can compute numerically the utility gain from entry for each point to the left of x_1. The picture shows the results.



Each citizen to the left of x_1 worsens the policy outcome after entry. The shape of the curve can be explained as follows. When the entrant is very left-wing, say $y = 0$, that will cause x_1 to prefer an alliance to the right. This will outweigh the gains obtained when y becomes formateur. This remains true as y moves right. This actually reduces the seats for x_1, meaning that whenever x_1 forms its coalition it will have to include parties which are farther to the right. At some point y becomes sufficiently close to x_1, so that whenever x_1 becomes formateur it will form a coalition $\{y, x_1\}$. This is the point at which the upward jump occurs. However, it still remains true that parties to the right of x_1 will become formateur with strictly positive probability and this will cause an expected utility loss for the left-wing entrant. As y gets closer to x_1 the gain decreases because there is little improvement in the policy outcome when the coalition $\{y, x_1\}$ is formed (y is close to x_1 so entry does not produce much of a policy gain) and it remains true that right-wing coalitions are formed when the formateur is sufficiently to the right. Since the change in utility due to entry is strictly negative for each citizen left of x_1, entry is not profitable whenever α is low enough.

2.5 Contiguous Coalitions

Some additional results can be obtained for equilibria in which all governing coalitions that are formed are 'contiguous'. By this we mean that whenever party j and party $i > j$ belong to a governing coalition then each party indexed $k \in (j, i)$ also belongs to the coalition. In other words, a formateur always chooses the closest parties to form a government. It is not obvious that it should be so. Consider for example a situation in which a left-wing party has 47 seats, a centrist party has 47 seats and a right-wing party has 6 seats. When the left-wing party is formateur it can choose between the centrist and the right-wing party. While the platform of the centrist party is closer to the one of the left-wing party, it is also implemented with probability 0.5. An alliance with the right-wing party instead yields the preferred policy with probability 47/53, i.e. 88.68%. Depending on the value of the parameters the 'un-natural' coalition between left and right might be preferable, for the left-wing party, to the 'natural' coalition with the centrist party.

While exceptions are possible, it is still interesting to characterize equilibria in which contiguous coalitions are always formed, since contiguous coalitions are so common. The next result provides some characterization for equilibria in which only contiguous coalitions are formed.

Proposition 4 *There is M^* such that whenever the number of seats is $M \geq M^*$ then any equilibrium $\{x_1, \ldots, x_n\}$ with no dominant party and with only contiguous coalitions is such that either $F(x_1) \leq \frac{1}{M}$ or party 1 is included in coalitions proposed by other parties. Furthermore, either $F(x_n) \geq \frac{M-1}{M}$ or party n is included in coalitions proposed by other parties.*

The proposition states that, unless the extreme parties at the left and the right are very extreme, they must end up being part of the governing coalitions proposed by more centrist parties. In other words, except for fringe parties, elections will typically produce at least a left-wing coalition including all parties on the center-left and a right-wing coalition including all parties on the center-right. Centrist coalitions (i.e. coalitions excluding *both* x_1 and x_n) are also possible, but center-left and center-right coalitions must occur with strictly positive probability. The logic is that an extreme left-wing (right-wing) party which is never chosen by other parties to form a coalition becomes vulnerable to entry to its left (right), since entry does not change adversely the equilibrium. It is only when the party is very extreme that this type of entry becomes unprofitable, since it does not yield seats. Thus, when coalitions are contiguous, very extreme parties (i.e. an extreme left party such that $F(x_1) \leq \frac{1}{M}$ or an extreme right party such that $F(x_n) \geq \frac{M-1}{M}$) can reach government only when they are chosen as formateurs. Typically, this will be an event with very small probability, since with free entry more parties can enter close to x_1 and x_n thus lowering the vote share of extremists.

3 Equilibria with Medium-Sized Parties

In this section we present an example to show how equilibria with medium–sized parties, i.e. parties which are not dominant but get a substantial number of seats, are possible. The main issue here is to explain how, absent the discontinuity related to dominant parties, a medium–sized party can avoid splitting into many smaller parties.

Splitting a medium-sized party may create new coalitional possibilities, negatively affecting the citizen who creates the new party. For example, consider a citizen voting for a medium–sized centrist party, and consider a voter to the left of the ideal point. When contemplating whether to create a new party, the voter faces the following trade–off:

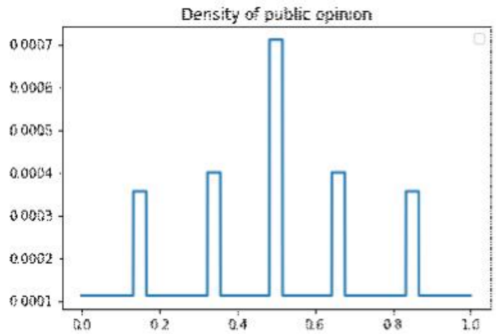
- on the positive side, the preferred point will be considered whenever the governing coalition ends up including the newly created party;
- on the negative side, parties on the right may now be able to form coalitions with the centrist party. Since the centrist party has lost seats in favor of the entrant, in such coalitions the centrist position will be weakened in favor of more right-wing positions.

If the second effect is sufficiently strong then in equilibrium medium–sized parties can survive. The argument is similar to the one in Indridason [20]. The difference is that in Indridason [20] the number and positions of the party are fixed and voting is strategic. This may lead some voters to choose parties which are farther way than their ideal point because in this way the resulting coalitions which are formed are more favorable. In this paper we make the assumption that, in large elections in which each single vote has essentially measure zero, voters are not strategic, while politicians are. It is the entry decision of politicians that generates the mechanisms against the split of medium-sized parties.

We now present an example in which medium-sized parties are formed in equilibrium. The example is as follows. Public opinion is distributed on the interval $[0,1]$ according to the following density function:

$$g(x) = \begin{cases} \frac{h_1-\alpha}{2\varepsilon} & \text{if} \quad a-\varepsilon < x < a+\varepsilon \\ \frac{h_2-\alpha}{2\varepsilon} & \text{if} \quad b-\varepsilon < x < b+\varepsilon \\ \frac{h_3-\alpha}{2\varepsilon} & \text{if} \quad 0.5-\varepsilon < x < 0.5+\varepsilon \\ \frac{h_2-\alpha}{2\varepsilon} & \text{if} \quad 1-b-\varepsilon < x < 1-b+\varepsilon \\ \frac{h_1-\alpha}{2\varepsilon} & \text{if} \quad 1-a-\varepsilon < x < 1-a+\varepsilon \\ \delta & \text{elsewhere} \end{cases}$$

where α, ε and δ are strictly positive and very small, h_1, h_2 and h_3 are strictly positive and such that $\int_0^1 g(x)\,dx = 1$. The distribution is symmetric, continuous and strictly positive at each point in $[0,1]$. The mass is concentrated at two extreme positions (a and $1-a$), two moderate positions (b and $1-b$) and a centrist position at the median. The following picture provides an example of such a density function.



There are $M = 100$ seats to be assigned. Consider a party system with 5 parties located at the peaks of the distribution, i.e. $x_1 = a$, $x_2 = b$, $x_3 = 0.5$, $x_4 = 1-b$ and $x_5 = 1-a$. Call $\{v_i\}_{i=1}^{5}$ the vote shares and notice that $v_1 = v_5$ and $v_2 = v_4$. We choose the values of h_1, h_2 and h_3 so that $v_2 + v_3 = 0.5$. Notice that this implies $v_1 = \frac{v_3}{2}$.

In Appendix II we show the conditions for such a party system to be an equilibrium and provide values of the parameters satisfying those conditions. In the equilibrium the parties located at x_1 and x_2 form a center-left coalition $\{x_1, x_2, x_3\}$, the party located at the median forms a centrist coalition $\{x_2, x_3, x_4\}$ and the parties located at x_4 and x_5 form a center-right coalition $\{x_3, x_4, x_5\}$. To get a sense of how the equilibrium works, consider entry by a new party located at $y < x_1$. In order to gain seats such an entry must occur close to x_1. When y becomes formateur it will form the coalition $\{y, x_1, x_2, x_3\}$. This is better than without entry, but limitedly so since y is close to x_1. On the other hand, x_2 can now form a coalition $\{x_1, x_2, x_3\}$ and exclude y. Thus the left-wing position loses weight whenever the center-left coalition is formed. This makes y worse off. The second effect is stronger, so entry is not profitable.

4 Thresholds

An important advantage of the framework discussed in this paper is that it makes it possible to evaluate the consequences of changes in the electoral formula. There are many variants of the proportional system. Often, the changes in the electoral formula are justified by the idea that they might promote stability in government and a limitation to the fragmentation of representation. In order to properly evaluate such claims we need a model in which entry is possible and we have to look at long-run equilibria. In other words, we have to take into account the incentives to exit and to enter induced by changes in the electoral formula, something that it is impossible to do in models with a fixed number of parties.

While a complete discussion of the impact of different variants of the proportional system is beyond the scope of this paper, we want to present a preliminary analysis of a frequently adopted modification of the 'pure' proportional system, i.e. minimum legal thresholds for representation. This means that a party can obtain seats only if it obtains at least a certain fraction of votes[5].

Proposition 5 *Suppose $c = 0$ and a threshold equal to v^* is imposed for representation, i.e. only parties receiving a vote share greater or equal to v^* are assigned seats. Then:*

1. *Any equilibrium of the pure proportional system (i.e., without threshold) in which each party obtains a share greater or equal to v^* remains an equilibrium.*
2. *Assume that $\frac{1}{v^*}$ is an integer number. Then there is always an equilibrium in which exactly $\frac{1}{v^*}$ parties enter.*

The logic is straightforward. Suppose, for example, that there are M seats and a threshold of $\frac{1}{M}$ is imposed for representation. Then there is one equilibrium (there may be others with fewer parties) in which exactly M parties enter, located in such a way that each party gets exactly a share $\frac{1}{M}$. At that point no further entry is possible, since an entrant would get strictly less than $\frac{1}{M}$ and thus no representation. Notice that the imposition of a minimum threshold of $\frac{1}{M}$ is not irrelevant. When there are M seats, there may be equilibria in which some parties get representation even if their vote share is below $\frac{1}{M}$, thanks to high rests. Those equilibria are eliminated by the imposition of a minimum threshold. For example, suppose that there are 10 seats, one party getting 55% of the vote and 5 parties getting 9% of the vote each. Then under the Largest Remainder rule the largest party gets 5 seats and each of the smaller parties gets 1 seat. This equilibrium would be destroyed by the introduction of the 10% threshold.

One interesting and counter-intuitive implication is the following. Suppose that a pure proportional system has only equilibria with fewer than M parties. Then imposing a threshold $\frac{1}{M}$, an apparently harmless modification, may actually *increase*

[5]Germany is a prominent example of a country using the proportional system with thresholds.

the number of parties. This will occur if, after the introduction of the threshold, the citizens play the equilibrium described in Proposition 5.

More in general, suppose that under a pure proportional system all equilibria are such that each party gets a share of the vote greater that $\frac{1}{M}$. Then the introduction of a threshold $\frac{1}{M}$ *expands* the set of equilibria, weakly increasing the number of parties which may be observed in equilibrium.

Another interesting observation is the following. The introduction of a threshold v^* can make it possible to have equilibria with parties at the most extreme positions, i.e. 0 and 1. Consider a situation in which the distribution F is such that, without the threshold, no robust equilibrium with a party located at 0 is possible; this is typically the case when F is such that entry at ε, for ε small enough, is profitable. This is no longer the case with the threshold. If the party located at 0 gets exactly v^* then entry at ε typically leads to a vote share lower than v^*. So, introduction of the threshold actually may support equilibria with more extremist parties

5 Literature Discussion and Extensions

In this section we first highlight the similarities and differences with other models of the proportional system which have been proposed in the literature and then discuss some possible variants of the model favoring the existence of equilibria with mid-sized parties.

5.1 Literature Discussion

Austen-Smith [1], Austen-Smith and Banks [2], and Baron and Diermeier [6] have analyzed models of proportional representation but only with a fixed (usually three) number of parties. The main question which these models try to answer is how likely it is that a coalition government be formed. These models are clearly not suitable for analyzing the question of how many parties will be formed in equilibrium when there is free entry and many different political positions may be held.

We are not aware of any work discussing the impact of variants of the proportional system, such as thresholds, from a theoretical point of view. Slinko and White [23] is an exception but they consider strategic voters whereas we assume sincere voting There is however a theoretical literature on general models of electoral rules, including proportional rules, and a specific literature on pure proportional systems. Cox [11], [12] produces a general analysis for the case of office–seeking parties looking at the properties of equilibria under many different systems, allowing in particular for multimember districts. His analysis however takes the number of parties as given and does not allow for free entry. This implies that the question 'how many viable parties does a certain electoral system induce' cannot be answered. Second, existence of equilibrium is not generally established and this makes some of the characterizations empty.

Two papers more closely related to ours are Hamlin and Hjortlund [18] and Bandyopadhyay and Oak [4]. Hamlin and Hjortlund also consider a citizen-candidate model in which citizens can freely form parties and then seats are distributed proportionally. There are some important differences between their model and ours. First, we adopt a different rule for policy formation. Hamlin and Hjortlund [18] assume that the policy implemented is the weighted average of the ideal points of the parties obtaining representation. We assume instead that the policy is a probability distribution over the ideal points of the parties in the governing coalition. This has very important consequences in terms of strategic behavior (see discussion in subsection 5.1.2). The second important difference is that we assume that political rents accrue to parties in proportion to the share of seats they obtain, while Hamlin and Hjortlund assume that rents are obtained only by the party with the highest vote share. This also implies very different strategic incentives. Under our assumption incentives to entry are much stronger, in particular for parties expecting a small share of the vote. This tendency to the proliferation of parties is widely considered an important characteristic of proportional representation[6]. At last, our results are obtained for general distributions, while most of their results are obtained for the uniform distribution.

5.1.1 Government vs. Legislative Rents

The main difference between Bandyopadhyay and Oak [4] and our paper is that they assume that rents are linked to government participation, while we assume that rents are linked to parliamentary participation. To see how the two hypotheses may lead to very different results, observe that in the Bandyopadhyay and Oak model there is an equilibrium in which only the citizen located at the median enters. No other candidate is willing to enter because it cannot affect the policy outcome and, since the entrant remains out of government, no political rents are obtained. This is not possible in our model, as long as entry costs are sufficiently low. If there is a single party located at the median then a new party entering close to the median can get almost 50% of the vote. Our assumption that rents are linked to parliamentary

[6] Another minor difference between our paper and Hamlin and Hjortlund [18] is that we assume a finite number of seats, so the share of seats may differ from the share of the vote.

representation implies that such an opportunity of creating a large party will not be overlooked.

5.1.2 Stochastic vs. Deterministic Policy Outcomes

We assume that no bargaining on policy is possible among the members of the governing coalition. Instead, the formation of a coalition leads automatically to a lottery. De Sinopoli and Iannantuoni (see [13] and [14]) consider an alternative set up in which the policy is determined as the weighted average of platform positions, as in Hamlin and Hjortlund [18] (see also Bandyopadhyay, Chatterjee and Sjöström [3] for a model allowing for both *ex ante* and *ex post* coalition formation). They show that this leads voters to choose strategically extremist parties. The emergence of extremist parties is the consequence on one hand of the particular way in which policy is formed and on the other hand of the strategic behavior of voters. This is easy to see in a simple example. Suppose there are three parties located at 0, 0.5 and 1 and that initially voters are sincere and the three parties are equally sized. In this case the policy selected is 0.5. Now suppose that a voter located, say, at 0.49 decides to vote strategically. It is clear that the voter is better off voting for the party located at 0. Similarly, a voter located at 0.51 is better off voting for the party located at 1. Thus, a centrist policy emerges as a compromise between extremist parties. If, instead, the policy implemented were a lottery over the extreme positions then the centrist voters would have no incentives to select extremist parties. Hamlin and Hjortlund [18] assume sincere voting, so these strategic incentives are not present.

Iaryczower and Mattozzi [19], in a model including equilibrium determination of campaign spending, assume that the policy is determined in way similar to this paper, i.e. as a probability distribution of the parties' platforms. The crucial difference is that they assign the weights according to the vote share, ignoring the problem of forming a governing coalition. They assume strategic voting, but if sincere voting were assumed then only maximum entry equilibria would be possible. The reason is that any citizen can increase in a favorable way her influence on policy by obtaining seats. The main entry-deterring mechanism, the risk of shifting in an undesired way the governing coalition, is absent in their model.

5.1.3 Strategic vs. Sincere Voting

In this paper we assume that voters are a continuum and potential candidates are a dense set of measure zero. Potential candidates behave strategically but voters choose mechanically the closest candidate. How good is the assumption of sincere voting in large elections? It is hard to provide empirical evidence about it (see Degan and Merlo [16] for a discussion). Given the current state of knowledge, it is worth exploring both models with sincere voting and models with strategic voting. One thing that should be pointed out is that strategic voting can easily be used to block entry and create equilibria with very few parties, using self-fulfilling expectations that any new entrant will not receive votes. Sincere voting makes it harder to prevent entry and to find equilibria with a limited number of parties. Models with strategic voting (such as Indridason [20], Cho [9], Cho [10] and Iaryczower and Mattozzi [19]) usually assume a fixed number of parties and do not discuss entry.

5.2 Variants Favoring Equilibria with Mid-Sized Parties

One of the goals of this paper is to show that, even without significant entry cost, proportional systems do not necessarily lead to a large number of parties. The main intuition is that, when candidates are policy-motivated, creating additional parties may affect adversely the formation of the governing coalitions. Here we want to discuss some modelling choices and the impact that they have on the existence of mid-sized parties.

Convex vs. linear disutility. Our model assumes single peaked preference of the form $v(x, y) = -|x - y|$, where y is the policy implemented and x is the ideal policy. A disutility linear in the distance is often used, but a frequently used alternative specification is quadratic disutility, $v(x, y) = -(x - y)^2$. This specification would make it easier to construct equilibria with mid-sized parties. The reason is that entering subtracts votes to the parties which are closest. This may improve policy when the entrant is selected but can worsen the policy when the formateur is far away. With quadratic utility the gains become smaller and the losses become larger.

The size of college districts. In this paper we have interpreted M as the number of seats in the parliament and we have assumed that the proportional election occurs in a single district. While this is actually the case for some countries (e.g. the Netherlands and Israel), most proportional system use multiple districts, each one with a limited and often variable number of seats. Smaller districts create an implicit threshold, whose effects may be different from those of a legal threshold at the national level. When a district is relatively small the possibilities of successful entry are reduced, so we expect the equilibrium number of parties to be smaller and mid-size parties to be more likely.

Formateur's choice. We assume that, absent a party with absolute majority, the probability of being formateur is equal to the vote share. Other rules are possible. Indridason [20] assumes that the formateur is always the party with the highest vote share. Adopting a rule like the one in Indridason [20] would favor mid-sized parties, since it penalizes in a stronger way the entry of candidates subtracting votes to the first party, providing an additional reason not to split. This effect will be true in general when the probability of being chosen as formateur increases more than proportionally in the vote share when a party becomes the most voted.

Higher cost of entry c. At last, an obvious way in which equilibria with medium sized parties may become more likely is through the increase of the participation cost c. Increases in c may occur either for economic factors (e.g. the relative wage of the kind of specialized labor needed to set up a party may increase) or through regulation (imposing burdensome requirements for putting on the ballot new parties). Other things equal, increases in c make it less likely that very small parties may survive. The opposite occurs when α goes up.

6 Conclusions

This paper discusses a model of the proportional system when the number of parties is endogenously determined. We have shown that in a citizen-candidate model equilibria with a relatively limited number of parties are possible, although when the number of seats is sufficiently large there will always be at least four parties. Our framework is flexible enough to allow us to discuss variants of the 'pure' proportional system, for example the introduction of thresholds for representation.

A natural next step would be to look at a dynamic models in which there is uncertainty on the distribution. Most dynamic models of elections have a fixed number of parties (a recent example is Baron [5]). Brusco and Roy [8] proposes a model in which entry and exit may occur in each period and the public opinion changes stochastically, but the electoral system they consider is first-past-the-post. The techniques could be in principle applied to proportional electoral systems[7], providing a theory of how equilibrium political configurations evolve.

[7] Trompounis and Xefteris [24] and De Sinopoli et al. [15] introduce uncertainty in a proportional model.

Appendix I

Proof of Proposition 1. If all citizens choose $\sigma_x = 0$ and no party enters then the utility is $\underline{u}_x$. A citizen located at x can form a party, obtain a seat share $s_x = 1$ and implement the preferred policy x, obtaining utility $\alpha - c > -1 \geq \underline{u}_x$. A profitable deviation exists and $\sigma_x = 0$ for each x is not an equilibrium.

Suppose $c < \alpha \widehat{s}_M$ and that only the citizen located at x^* chooses $\sigma_{x^*} = 1$, while all citizens with $x \neq x^*$ choose $\sigma_x = 0$. In any equilibrium with a single party it must be optimal for all other citizens to stay out. However, no matter where the entering party is located, there is some entrant who can get at least $\widehat{S}_M$ seats, while either not changing the policy or changing it in a favorable direction (this happens when the entrant ends up with at least as many seats as the incumbent). With $c < \alpha \widehat{s}_M$ this makes it optimal to enter, so there is no single party equilibrium. ■

Proof of Proposition 2. Let $\{x_1, \ldots, x_J\}$ be the locations of the parties entering in equilibrium, v_i the vote share obtained by the party located at x_i (called, for simplicity, party i) and S_i the number of seats. Let v_i^+ be the vote share collected by party i on the right of x_i, i.e. $v_i^+ = F\left(\frac{x_i + x_{i+1}}{2}\right) - F(x_i)$ if $i < J$ and $v_J^+ = 1 - F(x_J)$ otherwise. Let v_i^- be the vote share collected by party i on the left of x_i, i.e. $v_i^- = F(x_i) - F\left(\frac{x_i + x_{i-1}}{2}\right)$ if $i > 1$ and $v_1^- = F(x_1)$ otherwise. Assume that party k is dominant, so that $S_k \geq \left\lfloor \frac{M}{2} \right\rfloor + 1$.

<u>Claim 1</u>. For the dominant party $S_k \leq \left\lfloor \frac{M}{2} \right\rfloor + 2$.

This is trivial if $\left\lfloor \frac{M}{2} \right\rfloor + 2 \geq M$, which holds for $M \leq 4$. Assume $M > 4$ and suppose that there is an equilibrium in which the dominant party wins $S_k \geq \left\lfloor \frac{M}{2} \right\rfloor + 3$ seats. Suppose first $v_k^- \leq \frac{1}{M}$. Then entry immediately to the right of the dominant party yields at least $\left\lfloor \frac{M}{2} \right\rfloor + 1$. The reasoning is the same if $v_k^+ \leq \frac{1}{M}$. Thus, it must be $\min\left\{v_k^-, v_k^+\right\} > \frac{1}{M}$. Define $\overline{x}_k = \frac{x_k + x_{k+1}}{2}$ if $k < J$ and $\overline{x}_J = 1$, i.e. $\overline{x}_k$ is the citizen with the highest x among those who vote the dominant party k. Similarly, define $\underline{x}_k = \frac{x_{k-1} + x_k}{2}$ if $k > 1$ and $\underline{x}_1 = 0$, i.e. $\underline{x}_k$ is the citizen with the lowest x among those who vote the dominant party k. Suppose $k > 1$ and for each $y \in (x_{k-1}, x_k)$ define

$$v(y) = F\left(\frac{y + x_k}{2}\right) - F\left(\frac{x_{k-1} + y}{2}\right).$$

Since F is a continuous function, $v(y)$ is contiunous. Furthermore, $\lim_{y \uparrow x_k} v(y) = v_k^- > \frac{1}{M}$. If $\lim_{y \downarrow x_{k-1}} v(y) \geq \frac{1}{M}$ then entry just to the right of x_{k-1} yields at least one seat without changing the policy, since x_k remains the dominant party. If $\lim_{y \downarrow x_{k-1}} v(y) < \frac{1}{M}$ then continuity of $v(y)$ implies that there is a position y^* such that $v(y^*) = \frac{1}{M}$. Thus, entry at y^* yields exactly one seat and it does not change the policy, since at most one seat is subtracted from the seats of the dominant party.

Claim 2. A party not contiguous to k has at most two seats.

Consider a party not contiguous to the dominant party, i.e. its position is neither x_{k-1} nor x_{k+1}. For simplicity consider the party located at x_{k+2} (again, the argument for x_{k-2} is easily adapted). Suppose that the party obtains at least three seats. To achieve that it must get a vote share strictly greater than $\frac{2}{M}$. Thus, $\max\left\{v_{k+2}^-, v_{k+2}^+\right\} > \frac{1}{M}$. But this implies that a new party can enter either just to the right or just to the left of x_{k+2} and gain at least one seat without changing the policy. Thus any party which is not contiguous to the dominant party can't have more than two seats.

Claim 3. If a party contiguous to the dominant one has at least 3 seats, party k has exactly $\lfloor \frac{M}{2} \rfloor + 1$ seats.

Suppose that party $k+1$ has strictly more than two seats. Then its share of the vote must be strictly larger that $\frac{2}{M}$. If $v_{k+1}^+ > \frac{1}{M}$ then a party can enter just to the right of $k+1$ and gain at least one seat without changing the policy outcome. Thus, it must be that $v_{k+1}^+ \leq \frac{1}{M}$ and therefore $v_{k+1}^- > \frac{1}{M}$. If a contiguous party has at least 3 seats then there is a point $y^* \in (x_k, x_{k+1})$ such that

$$F\left(\frac{y^* + x_k}{2}\right) - F\left(\frac{x_k + x_{k-1}}{2}\right) = \frac{\lfloor \frac{M}{2} \rfloor + 1}{M}.$$

Thus, a party entering at any point $y \in (y^*, x_{k+1})$ would not change the policy. For y sufficiently close to x_{k+1} the entrant would gain at least one seats and we have a contradiction. We conclude that whenever party $k+1$ has more than two seats then k must have exactly $\lfloor \frac{M}{2} \rfloor + 1$ seats. The same reasoning applies to party $k-1$. ■

Proof of Proposition 3. Consider a configuration $\{x_1, x_2, x_3\}$ and assume that there is a continuum of seats. Let $\{v_1, v_2, v_3\}$ be the vote shares (coinciding with the seat shares) of the three parties.

Case 1. $\max\{v_1, v_2, v_3\} > 0.5$. If the dominant party is x_1 then entry in the interval (x_2, x_3) or $(x_3, 1]$ does not change the policy outcome and yields seats, so a profitable deviation is found. A similar reasoning holds if the dominant party is x_3. If the dominant party is x_2 then it must be $x_1 = 0$ and $x_3 = 1$, otherwise entry to the left of x_1 or right of x_3 would be profitable. But in that case entry at ε or $1-\varepsilon$, for ε small enough, becomes profitable.

Case 2. $\max\{v_1, v_2, v_3\} = 0.5$. First observe that $v_1 = 0.5$ and $v_3 = 0.5$ cannot be part of a robust equilibrium because a small movement to the left or to the right would give absolute majority. If $v_2 = 0.5$ then party 2 must be part of all coalitions and $v_1 + v_3 = 0.5$. The best option for x_1 is to form a coalition with x_2. The same it true for x_3. Suppose x_2 chooses x_1 (the case in which x_2 chooses x_3 is symmetric). Consider entry just to the left of x_3, i.e. at a position $y \in (x_3 - \varepsilon, x_3)$. Without entry, the expected disutility of a citizen located at y is

$$D_1(y) = v_1(y - x_1) + v_2(y - x_2) + v_3\left(\frac{v_2}{v_2+v_3}(y - x_2) + \frac{v_3}{v_2+v_3}(x_3 - y)\right).$$

In case of entry the vote share of x_1 does not change while the vote shares of the other parties change to

$$v_2' = F\left(\frac{x_2+y}{2}\right) - F\left(\frac{x_2+x_3}{2}\right) \qquad v_y = F\left(\frac{x_3+y}{2}\right) - F\left(\frac{x_2+y}{2}\right) \tag{2}$$

$$v_3' = 1 - F\left(\frac{x_3+y}{2}\right). \tag{3}$$

If entry causes x_2 to prefer an alliance with y then entry is clearly profitable. So, suppose x_2 still allies with x_1. The expected disutility for y in case of entry is

$$\begin{aligned} D_2(y) &= v_1(y - x_1) + v_2'(y - x_2) \\ &\quad + (v_y + v_3')\left(\frac{v_2'}{v_2'+v_y+v_3'}(y - x_2) + \frac{v_3'}{v_2'+v_y+v_3'}(x_3 - y)\right). \end{aligned}$$

Define

$$\Delta(y) = D_1(y) - D_2(y).$$

Entry is profitable if $\Delta(y) > 0$ for some $y \in (x_3 - \varepsilon, x_3)$. We have

$$\begin{aligned} \Delta(y) &= (v_2 - v_2')(y - x_2) + \frac{v_3}{1-v_1}(v_2(y - x_2) + v_3(x_3 - y)) \\ &\quad - \frac{(v_y + v_3')}{1-v_1}(v_2'(y - x_2) + v_3'(x_3 - y)) \end{aligned}$$

Notice that at $y = x_3$ we have $\Delta(x_3) = 0$. Thus, entry is profitable if at $y = x_3$ the left derivative of $\Delta(y)$ is negative. This is given by

$$\begin{aligned}
\frac{d^{-}\Delta(y)}{dy} &= -\frac{dv_2'}{dy}(y-x_2)+\left(v_2-v_2'\right)+\frac{v_3}{1-v_1}(v_2-v_3) \\
&\quad -\frac{1}{1-v_1}\left(\frac{d\left(v_y+v_3'\right)}{dy}\left(v_2'(y-x_2)+v_3'(x_3-y)\right)\right) \\
&\quad -\frac{1}{1-v_1}\left(\left(v_y+v_3'\right)\left(\frac{dv_2'}{dy}(y-x_2)+v_2'+\frac{dv_3'}{dy}(x_3-y)-v_3'\right)\right)
\end{aligned}$$

Now observe

$$\begin{aligned}
\lim_{y\uparrow x_3}\frac{d^{-}\Delta(y)}{dy} &= -\frac{dv_2'}{dy}(x_3-x_2)+\frac{v_3}{1-v_1}(v_2-v_3) \\
&\quad -\frac{1}{1-v_1}\left(\frac{d\left(v_y+v_3'\right)}{dy}v_2(x_3-x_2)\right) \\
&\quad -\frac{v_3}{1-v_1}\left(\left(\frac{dv_2'}{dy}(x_3-x_2)+v_2-v_3^{+}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= -\left(\frac{dv_2'}{dy}+\frac{v_3}{1-v_1}\frac{d\left(v_y+v_3'\right)}{dy}v_2+\frac{v_3}{1-v_1}\frac{dv_2'}{dy}\right)(x_3-x_2) \\
&\quad +\frac{v_3}{1-v_1}\left(v_3^{+}-v_3\right)
\end{aligned}$$

where $v_3^{+}=1-F(x_3)$ is the right constituency of x_3. Using

$$\begin{aligned}
\frac{d\left(v_y+v_3'\right)}{dy} &= \frac{d\left(1-F\left(\frac{x_2+y}{2}\right)\right)}{dy}=-\frac{1}{2}f\left(\frac{x_2+y}{2}\right) \\
\frac{dv_2'}{dy} &= \frac{d\left(F\left(\frac{x_2+y}{2}\right)-F\left(\frac{x_1+x_2}{2}\right)\right)}{dy}=\frac{1}{2}f\left(\frac{x_2+y}{2}\right)
\end{aligned} \tag{4}$$

We conclude

$$\lim_{y\uparrow x_3}\frac{d^{-}\Delta(y)}{dy}=-\frac{1}{2}f\left(\frac{x_2+x_3}{2}\right)(x_3-x_2)+\frac{v_3}{1-v_1}\left(v_3^{+}-v_3\right)<0.$$

since $v_3^{+}\leq v_3$.

<u>Case 3. $\max\{v_1,v_2,v_3\}<0.5$.</u> In this case every two-party coalition must have a majority of the seats. Suppose that party 2 forms a coalition $\{x_1,x_2\}$ when selected as formateur (the case in which party 2 forms a coalition $\{x_2,x_3\}$ is symmetric). We have 4 subsubcases.

<u>**Subcase 1**. Both x_1 and x_3 choose x_2.</u> This case is identical to Case 2.

<u>**Subcase 2**. x_1 chooses x_2 and x_3 chooses x_1</u>. First observe that, since x_3 prefers x_1 to x_2 it must be:

$$\frac{v_1}{v_1+v_3}\left(x_3-x_1\right)<\frac{v_2}{v_2+v_3}\left(x_3-x_2\right)<x_3-x_2. \tag{5}$$

Consider entry at $y=x_3-\varepsilon$ for $\varepsilon>0$ sufficiently small. Suppose x_2 still prefers x_1 after entry (if it forms a coalition with y after entry then this makes the entrant clearly better off). The disutility for citizen y without entry is:

$$\begin{aligned} D_1\left(y\right) &= v_1\left(y-x_1\right)+v_2\left(y-x_2\right) \\ &\quad +v_3\left(\frac{v_1}{1-v_2}\left(y-x_1\right)+\frac{v_3}{1-v_2}\left(x_3-y\right)\right). \end{aligned}$$

In case of entry disutility becomes

$$\begin{aligned} D_2\left(y\right) &= v_1\left(y-x_1\right)+v_2'\left(y-x_2\right) \\ &\quad +\left(v_y+v_3'\right)\left(\frac{v_1}{1-v_2'}\left(y-x_1\right)+\frac{v_3'}{1-v_2'}\left(x_3-y\right)\right) \end{aligned}$$

Let $\Delta\left(y\right)=D_1\left(y\right)-D_2\left(y\right)$. As in Case 2, we have $\Delta\left(x_3\right)=0$ and we want to show $\lim_{y\uparrow x_3}\frac{d\Delta(y)}{dy}<0$. We have

$$\lim_{y\uparrow x_3}\frac{dD_1\left(y\right)}{dy}=v_1+v_2+v_3\frac{v_1-v_3}{1-v_2}$$

$$\begin{aligned} \lim_{y\uparrow x_3}\frac{dD_2\left(y\right)}{dy} &= v_1+\frac{dv_2'}{dy}\left(x_3-x_2\right)+v_2+\frac{d\left(v_y+v_3'\right)}{dy}\left(\frac{v_1}{1-v_2}\left(x_3-x_1\right)\right) \\ &\quad +v_3\left(\frac{d}{dy}\left(\frac{v_1}{1-v_2'}\right)\left(x_3-x_1\right)+\frac{v_1}{1-v_2}\right)-v_3\left(\frac{v_3^+}{1-v_2}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{y\uparrow x_3}\frac{d\Delta\left(y\right)}{dy} &= v_3\frac{v_3^+-v_3}{1-v_2}-\frac{dv_2'}{dy}\left(x_3-x_2\right) \\ &\quad -\frac{d\left(v_y+v_3'\right)}{dy}\left(\frac{v_1}{1-v_2}\left(x_3-x_1\right)\right)-v_3\left(\frac{d}{dy}\left(\frac{v_1}{1-v_2'}\right)\left(x_3-x_1\right)\right) \end{aligned}$$

Using (4) and

$$\frac{d}{dy}\left(\frac{v_1}{1-v_2'}\right)=\frac{v_1}{\left(1-v_2'\right)^2}\frac{1}{2}f\left(\frac{x_2+y}{2}\right).$$

We conclude:

$$\lim_{y\uparrow x_3}\left(\frac{dD_1(y)}{dy}-\frac{dD_2(y)}{dy}\right)=v_3\frac{v_3^+-v_3}{1-v_2}-\frac{1}{2}f\left(\frac{x_2+x_3}{2}\right)(x_3-x_2)$$

$$+\frac{1}{2}f\left(\frac{x_2+x_3}{2}\right)\left(\frac{v_1}{1-v_2}(x_3-x_1)\right)-v_3\left(\frac{v_1}{(1-v_2)^2}\frac{1}{2}f\left(\frac{x_2+x_3}{2}\right)(x_3-x_1)\right)$$

Inequality (5) implies

$$\frac{1}{2}f\left(\frac{x_2+y}{2}\right)\left(\frac{v_1}{1-v_2}(x_3-x_1)\right)<\frac{1}{2}f\left(\frac{x_2+x_3}{2}\right)(x_3-x_2)$$

so we can conclude $\lim_{y\uparrow x_3}\frac{d\Delta(y)}{dy}<0$ and profitable entry exists.

Subcase 3. x_1 chooses x_3 and x_3 chooses x_1. Since x_1 prefers x_3 to x_2 it must be the case that

$$\frac{v_3}{v_1+v_3}(x_3-x_1)<\frac{v_2}{v_1+v_2}(x_2-x_1).$$

Since x_3 prefers x_1 to x_2 it must be the case that

$$\frac{v_1}{v_1+v_3}(x_3-x_1)<\frac{v_2}{v_2+v_3}(x_3-x_2)$$

Let $q=\max\left\{\frac{v_2}{v_1+v_2},\frac{v_2}{v_2+v_3}\right\}$. Summing side by side we obtain:

$$\begin{aligned}x_3-x_1 &< \frac{v_2}{v_1+v_2}(x_2-x_1)+\frac{v_2}{v_2+v_3}(x_3-x_2)\\ &< q(x_2-x_1)+q(x_3-x_2)<q(x_3-x_1)\end{aligned}$$

which is impossible because $q<1$. Thus, this case can be ignored.

Subcase 4. x_1 chooses x_3 and x_3 chooses x_2. Since x_1 prefers x_3 to x_2 this implies

$$\frac{v_3}{v_1+v_3}(x_3-x_1)<\frac{v_2}{v_1+v_2}(x_2-x_1) \tag{6}$$

In turn, this implies

$$\frac{v_3}{v_1+v_3}(x_3-x_1)<\frac{v_2}{v_1+v_2}(x_3-x_1)\qquad\longrightarrow\qquad v_3<v_2$$

Since x_2 prefers x_1 to x_3 it must be

$$\frac{v_1}{v_1+v_2}(x_2-x_1)<\frac{v_3}{v_2+v_3}(x_3-x_2) \tag{7}$$

and since x_3 prefers x_2 to x_1 it must be

$$\frac{v_2}{v_2+v_3}(x_3-x_2) < \frac{v_1}{v_1+v_3}(x_3-x_1). \tag{8}$$

Summing (6) and (7) side by side we have

$$\frac{v_3}{v_1+v_3}(x_3-x_1) + \frac{v_1}{v_1+v_2}(x_2-x_1) <$$

$$\frac{v_2}{v_1+v_2}(x_2-x_1) + \frac{v_3}{v_2+v_3}(x_3-x_2).$$

Afterm manipulations we get

$$\left(\frac{v_3}{v_1+v_3} - \frac{v_3}{v_2+v_3}\right)(x_3-x_1) < \left(\frac{v_2-v_1}{v_1+v_2} - \frac{v_3}{v_2+v_3}\right)(x_2-x_1)$$

and

$$v_3\left(\frac{v_2-v_1}{(v_1+v_3)}\right)(x_3-x_1) < \left(\frac{(v_2-v_1)\,v_2 - 2v_1v_3}{(v_1+v_2)}\right)(x_2-x_1) \tag{9}$$

If $v_1 < v_2$ then we have

$$v_1(v_2+v_3)(v_1-v_2+2v_3) < 0$$

which is impossible because $v_1 + 2v_3 > \frac{1}{2} > v_2$.

Now suppose $v_1 \geq v_2 \geq v_3$. If $x_2 - x_1 \geq x_3 - x_2$ then (7) implies

$$\frac{v_1}{v_1+v_2} < \frac{v_3}{v_2+v_3} \qquad \longrightarrow \qquad v_1 < v_3,$$

a contradiction. Then it must be $x_3 - x_2 > x_2 - x_1$. Rewrite inequalities (6) and (7) as

$$(x_3-x_1) < \frac{v_1+v_3}{v_3}\frac{v_2}{v_1+v_2}(x_2-x_1)$$

$$(x_2-x_1) < \frac{v_1+v_2}{v_1}\frac{v_3}{v_2+v_3}(x_3-x_2).$$

Combining the two

$$(x_3-x_1) < \frac{v_1+v_3}{v_3}\frac{v_2}{v_1+v_2}\frac{v_1+v_2}{v_1}\frac{v_3}{v_2+v_3}(x_3-x_2)$$

which implies

$$1 < \frac{x_3-x_1}{x_3-x_2} < \frac{v_2(v_1+v_3)}{v_1(v_2+v_3)} \quad \rightarrow \quad v_3(v_1-v_2) < 0.$$

which is impossible because $v_1 \geq v_2$. ■

Proof of Proposition 4. Suppose that $F(x_1) > \frac{1}{M}$ and x_1 participates to a governing coalition only when x_1 is picked as formateur. Let k be the highest integer such that $\frac{k}{M} < F(x_1)$. Since $F(x_1) > \frac{1}{M}$ then $k \geq 1$. Let y be such that $F\left(\frac{y+x_1}{2}\right) = \frac{k}{M}$. Then the party positioned at y obtains exactly k seats by entering. The vote share of the entrant does not generate any rest, so the parties located at $\{x_2, \ldots, x_n\}$ get exactly the same seats as before while x_1 obtains exactly k fewer seats. Entry is unprofitable only if it changes adversely the policy outcome. The parties located at $\{x_2, \ldots, x_n\}$ have the same seats as before entry. If each one of them was forming a coalition government without x_1, whatever they were doing before entry by y is still feasible. Forming coalitions including both y and x_1 for parties that were excluding x_1 from their coalitions is clearly dominated by forming the same coalitions formed before entry. Thus, any change must include the party at x_1. Since all the coalitions are contiguous, this can only happens if x_1 substitutes some party x_k on the far right of the coalition or x_1 is added to the coalition. In both cases the utility of y is strictly increased. Finally, y is better off when it is the formateur. When x_1 becomes formateur it will include y in the coalition as long as y is sufficiently close to x_1, which will be the case when the density f is strictly positive and continuous on $[0, 1]$ and M is large enough. We conclude that entry is strictly profitable for y, contradicting the claim that $\{x_1, \ldots, x_n\}$ is an equilibrium configuration. The reasoning for x_n is symmetric. ■

Proof of Proposition 5. The first point is trivial. If a configuration $\{x_1, \ldots, x_k\}$ is an equilibrium under a pure proportional system then no other party can profitably enter. This remains true when the threshold is imposed. Similarly, if each existing party has a share of the vote greater than v^*, then the introduction of the threshold does not affect the outcome and thus it cannot make it profitable to exit.

Consider now the second point. Let $\frac{1}{v^*}$ be integer. Consider an equilibrium in which $n^* = \frac{1}{v^*}$ parties enter and their locations are obtained as a solution to the system of equations

$$F\left(\frac{x_i + x_{i+1}}{2}\right) - F\left(\frac{x_i + x_{i-1}}{2}\right) = v^*, \tag{10}$$

with $i = 1, \ldots, n^*$ and the conventions $F\left(\frac{x_1+x_0}{2}\right) = 0$, $F\left(\frac{x_{n^*}+x_{n^*+1}}{2}\right) = 1$. Each party obtains exactly a share v^* of the vote and it obtains representation. No party is better off exiting, since this would cause a loss of rents and would change unfavorably the lottery on the implemented policy. No entrant can obtain representation: Any entering party would obtain strictly less than $v^*\%$ of the votes, and therefore no representation, and it would cause the closest parties to lose representation, thus affecting negatively the policy. Thus, the proposed configuration is an equilibrium. ■

Appendix II

A A Non-Robust Equilibrium with 3 Parties

To avoid cumbersome notation assume that there is a continuum of seats so that the percentage of seats is equal to the percentage of votes. Since no party has a majority, the outcome of the electoral process is the following:

- L gets to form the governing coalition with probability $\frac{1}{3}$. When this happens the governing coalition is $\{L, C\}$ and each one of the two parties gets the preferred point with probability $\frac{1}{2}$.
- C gets to form the governing coalition with probability $\frac{1}{3}$. Since C is indifferent between L and R, the coalitions $\{L, C\}$ and $\{C, L\}$ are chosen with equal probability. Whenever a coalition is formed, each one of the two parties gets the preferred point with probability $\frac{1}{2}$.
- R gets to form the governing coalition with probability $\frac{1}{3}$. When this happens the governing coalition is $\{C, R\}$ and each one of the two parties gets the preferred point with probability $\frac{1}{2}$.

From an *ex ante* point of view the probability distribution on outcomes is L with probability $\frac{1}{4}$, C with probability $\frac{1}{2}$ and R with probability $\frac{1}{4}$. It is clear that C prefers to stay rather then leave. If L were to leave then C would have an absolute majority. Thus, the condition for it being optimal to stay is

$$C - L \geq \frac{1}{2}(C - L) + \frac{1}{4}(R - L)$$

which is satisfied with equality because $C - L = R - C$. The same is true for R.

Consider now the incentives to enter. We will check that there are no incentives to enter for parties located at $x < C$, the case $x > C$ is symmetric.

Suppose first $x < L$. In this case the entrant gets a share of the vote

$$v_x = \frac{4}{3}\left(\frac{x + L}{2}\right) = \frac{2}{3}x + \frac{1}{6}$$

and the party located at L gets $\frac{1}{3} - v_x$. Since the entrant gets a share of the vote of at least $\frac{1}{6}$, a coalition between L and C is no longer a governing coalition. This implies that C will always form a coalition with R, since the alternative is to form a coalition with $\{x, L\}$ and this is less preferred (C is indifferent between L and R but strictly prefers R to x). Thus, if x enters, the coalition $\{x, L, C\}$ will be formed whenever x or L are selected and the coalition $\{C, R\}$ will be formed whenever C or R are selected. Since the total vote share of x and L remains $\frac{1}{3}$, the expected value in case of entry for a candidate located at $x < L$ is

$$\frac{1}{3}\left(\frac{\frac{1}{3}-v_x}{\frac{2}{3}}(L-x)+\frac{1}{2}(C-x)\right)+\frac{2}{3}\left(\frac{1}{2}(C-x)+\frac{1}{2}(R-x)\right)=$$

$$\frac{1-3v_x}{6}(L-x)+\frac{1}{2}(C-x)+\frac{1}{3}(R-x)$$

So entry is not profitable if

$$\frac{1-3v_x}{6}(L-x)+\frac{1}{2}(C-x)+\frac{1}{3}(R-x)\geq$$

$$\frac{1}{4}(L-x)+\frac{1}{2}(C-x)+\frac{1}{4}(R-x)$$

or

$$R-L\geq 3v_x(L-x)$$

which is always satisfied.

Entry in the interval $(L, C-2\varepsilon]$ yields a vote share of zero, so it can be ignored. Consider now entry in the interval $(C-2\varepsilon, C)$. Upon entry, the vote shares are

$$\begin{aligned}
v_L &= \frac{1}{3} \\
v_x &= \frac{1}{6\varepsilon}\left(\frac{x+C}{2}-(C-\varepsilon)\right)=\frac{1}{6}-\frac{C-x}{12\varepsilon} \\
v_C &= \frac{1}{6\varepsilon}\left(C+\varepsilon-\frac{x+C}{2}\right)=\frac{1}{6}+\frac{C-x}{12\varepsilon} \\
v_R &= \frac{1}{3}
\end{aligned}$$

As a consequence of entry, C may no longer be indifferent between R and L. In particular, C is going to prefer an alliance with $\{L, x\}$ to an alliance with R if

$$\frac{v_R}{v_R+v_C}(R-C)\geq\frac{v_L}{v_L+v_x+v_C}(C-L)+\frac{v_x}{v_L+v_x+v_C}(C-x)$$

Using the expressions for the vote shares, this inequality can be written as:

$$\frac{2}{3+\frac{C-x}{2\varepsilon}}(R-C) \geq \frac{1}{2}(C-L)+\left(\frac{1}{4}-\frac{C-x}{8\varepsilon}\right)(C-x)$$

and using $R-C=C-L$ we obtain

$$\left(\frac{1}{C-x+6\varepsilon}\right)(R-C)+\frac{1}{4\varepsilon}(x-C) \geq 0. \tag{11}$$

Since $\varepsilon < \frac{1}{50}$, the inequality is satisfied at $x = C-2\varepsilon$. The expression is increasing in x, so this a sufficient condition to have inequality 11 satisfied for each $x \in (C-2\varepsilon, C)$.

B An Equilibrium with Medium-Sized Parties

The parties are located at $\{a, b, 0.5, 1-b, 1-a\}$ and there are 100 seats. We choose the parameters $\{\delta, h_1, h_2, h_3\}$ of the density function so that $v_i = s_i$ for each party (no rests) and $s_2+s_3 = 0.5$. Notice that, given the symmetry of the distribution, we have $s_2 = s_4$ and $s_1 = s_5$. These conditions, together with $\sum_{i=1}^{5} s_i = 1$, imply $2s_1 = s_3$.

The positions a and b are chosen so that the parties will form the coalitions as follows:

- When x_1 and x_2 are formateurs, which happens with probability $v_1 + v_2$, the coalition $\{x_1, x_2, x_3\}$ is formed.
- When x_3 is formateur, which happens with probability v_3, the coalition $\{x_2, x_3, x_4\}$ is formed.
- When x_4 and x_5 are formateurs, which happens with probability $v_4 + v_5$, the coalition $\{x_3, x_4, x_5\}$ is formed.

The coalition choices of x_1, x_3 and x_5 are obviously optimal. For x_2 or x_4 (the center-left and the center-right parties) optimality is not obvious, as a centrist coalition $\{x_2, x_3, x_4\}$ could be better. To make sure that x_2 prefer a center-left to a centrist coalition the following condition needs to be satisfied:

$$\frac{v_1}{v_1+0.5}(b-a)+\frac{v_3}{v_1+0.5}(0.5-b) < \frac{v_3}{v_2+0.5}(0.5-b)+\frac{v_4}{v_2+0.5}(1-2b)$$

which, after manipulations, reduces to

$$\frac{v_1}{v_1+v_2} < \frac{0.5-b}{b-a} \tag{12}$$

A symmetric condition holds for x_4. The parameter will be chosen to satisfy condition 12.

B.1 No Incentives to Exit

We first check that no party wants to exit. Given the symmetry of the equilibrium we need to do it only for the parties located at x_1, x_2 and x_3. Let $U_C^{x^*}$ be the expected utility of a party located at x^* when the coalition C is formed.

B.1.1 No Exit for the Party Located at a

For the party located at $x_1 = a$ the utility of staying is

$$U^{\text{in}}(x_1) = (v_1+v_2)\, U^{x_1}_{\{x_1,x_2,x_3\}} + v_3 U^{x_1}_{\{x_2,x_3,x_4\}} + (v_4+v_5)\, U^{x_1}_{\{x_3,x_4,x_5\}}$$

$$= (v_1+v_2)\left(U^{x_1}_{\{x_1,x_2,x_3\}} + U^{x_1}_{\{x_3,x_4,x_5\}}\right) + v_3 U^{x_1}_{\{x_2,x_3,x_4\}}$$

where we used $v_1+v_2 = v_4+v_5$. The utilities $U_C^{x_1}$ are defined as follows.

$$\begin{aligned}
U^{x_1}_{\{x_1,x_2,x_3\}} &= -\left(\frac{v_2}{v_1+v_2+v_3}(b-a) + \frac{v_3}{v_1+v_2+v_3} \times (0.5-a)\right) \\
&= -\left(\frac{0.5}{v_1+0.5}(b-a) + \frac{v_3}{v_1+0.5}(0.5-b)\right)
\end{aligned}$$

where we used $v_2+v_3 = 0.5$. Similarly, we have:

$$\begin{aligned}
U^{x_1}_{\{x_2,x_3,x_4\}} &= -\left(\frac{v_2}{v_2+0.5}(b-a) + \frac{v_3}{v_2+0.5}(0.5-a) + \frac{v_4}{v_2+0.5}(1-b-a)\right) \\
&= -\left(\frac{2v_2}{v_2+0.5}(0.5-a) + \frac{v_3}{v_2+0.5}(0.5-a)\right) = -(0.5-a)
\end{aligned}$$

$$\begin{aligned}
U^{x_1}_{\{x_3,x_4,x_5\}} &= -\left(\frac{v_3}{v_5+0.5}(0.5-a) + \frac{v_4}{v_5+0.5}(1-b-a) + \frac{v_5}{v_5+0.5}(1-2a)\right) \\
&= -\left(\frac{2v_1+0.5}{v_1+0.5}(0.5-a) + \frac{v_2}{v_1+0.5}(0.5-b)\right)
\end{aligned}$$

Notice that

$$U^{x_1}_{\{x_1,x_2,x_3\}} + U^{x_1}_{\{x_3,x_4,x_5\}} = -2\,(0.5 - a)$$

We conclude

$$U^{\text{in}}_{x_1} = -\,(0.5 - a) \times (2\,(v_1 + v_2) + v_3) = -\,(0.5 - a)$$

If the party located at x_1 were to exit, the party located at x_2 would receive a share $v_1 + v_2$ of the vote and there would be no other changes. The policy outcomes would be as follows:

- When x_2 and x_3 are formateurs, which happens with probability $v_1 + 0.5$, the coalition $\{x_2, x_3\}$ is formed.
- When x_4 and x_5 are formateur, which happens with probability $v_1 + v_2$, the coalition $\{x_3, x_4, x_5\}$ is formed.

Thus, in case of exit, the utility of leaving for the party located at x_1 is

$$U^{\text{out}}_{x_1} = (v_1 + 0.5)\,\widehat{U}^{x_1}_{x_2,x_3} + (v_1 + v_2)\,U^{x_1}_{x_3,x_4,x_5}$$

where

$$\widehat{U}^{x_1}_{x_2,x_3} = -\left(\frac{v_1 + v_2}{v_1 + 0.5}\,(b - a) + \frac{v_3}{v_1 + 0.5}\,(0.5 - a)\right)$$

So

$$U^{\text{out}}_{x_1} = -\,(v_1 + v_2)\,(b - a) - v_3\,(0.5 - a) + (v_1 + v_2)\,U^{x_1}_{x_3,x_4,x_5}$$

Thus, the condition $U^{\text{in}}_{x_1} \geq U^{\text{out}}_{x_1}$ is equivalent to

$$(b - a)\,v_1 \geq (0.5 - b)\,v_3.$$

Since $v_1 = \frac{1}{2}v_3$ this reduces to

$$\frac{1}{2} \geq \frac{0.5 - b}{b - a}. \tag{13}$$

Thus, the parameters a and b must be chosen so that inequality 13 is satisfied.

B.1.2 No Exit for the Party Located at b

For the party located at $x_2 = b$ the utility of staying is

$$\begin{aligned} U^{\text{in}}\,(x_2) &= (v_1 + v_2)\,U^{x_2}_{\{x_1,x_2,x_3\}} + v_3 U^{x_2}_{\{x_2,x_3,x_4\}} + (v_4 + v_5)\,U^{x_2}_{\{x_3,x_4,x_5\}} \\ &= (v_1 + v_2)\left(U^{x_2}_{\{x_1,x_2,x_3\}} + U^{x_2}_{\{x_3,x_4,x_5\}}\right) + v_3 U^{x_2}_{\{x_2,x_3,x_4\}} \end{aligned}$$

where

$$U^{x_2}_{\{x_1,x_2,x_3\}} = -\left(\frac{v_1}{v_1+0.5}\times(b-a)+\frac{v_3}{v_1+0.5}\times(0.5-b)\right)$$

$$U^{x_2}_{\{x_2,x_3,x_4\}} = -\left(\frac{v_3}{v_2+0.5}\times(0.5-b)+\frac{v_4}{v_2+0.5}\times(1-2b)\right) = -(0.5-b)$$

:

$$\begin{aligned} U^{x_2}_{\{x_3,x_4,x_5\}} &= -\left(\frac{v_3}{v_5+0.5}\times(0.5-b)+\frac{v_4}{v_5+0.5}\times(1-2b)+\frac{v_5}{v_5+0.5}\times(1-a-b)\right) \\ &= -\left(\frac{1}{v_1+0.5}\times(0.5-b)+\frac{v_1}{v_1+0.5}\times(b-a)\right) \end{aligned}$$

Then

$$U^{x_2}_{\{x_1,x_2,x_3\}} + U^{x_2}_{\{x_3,x_4,x_5\}} = -\left(2\times(0.5-b)+\frac{2v_1}{v_1+0.5}\times(b-a)\right)$$

and we conclude

$$U^{\text{in}}_{x_2} = -(v_1+v_2)\left(2\times(0.5-b)+\frac{2v_1}{v_1+0.5}\times(b-a)\right) - v_3(0.5-b)$$

$$= -(0.5-b) - \frac{(1-2v_1)\,v_1}{v_1+0.5}(b-a)\,.$$

In case of exit the vote share of x_2 goes to x_3, who gets 50%. The policy outcomes are as follows:

- When x_1 is formateur, which happens with probability v_1, the coalition $\{x_1, x_3\}$ is formed.
- When x_3 and x_4 are formateurs, which happens with probability $0.5 + v_2$, the coalition $\{x_3, x_4\}$ is formed.
- When x_5 is formateur, which happens with probability v_1, the coalition $\{x_3, x_4, x_5\}$ is formed.

The expected utility in case of exit is therefore

$$U^{\text{out}}_{x_2} = v_1\widehat{U}^{x_2}_{\{x_1,x_3\}} + (0.5+v_2)\,\widehat{U}^{x_2}_{\{x_3,x_4\}} + v_1\widehat{U}^{x_2}_{\{x_3,x_4,x_5\}}$$

where

$$\widehat{U}^{x_2}_{\{x_1,x_3\}} = -\left(\frac{v_1}{v_1+0.5}(b-a)+\frac{0.5}{v_1+0.5}(0.5-b)\right)$$

$$\begin{aligned}\widehat{U}^{x_2}_{\{x_3,x_4\}} &= -\left(\frac{0.5}{v_2+0.5}(0.5-b)+\frac{v_2}{v_2+0.5}(1-2b)\right)\\ &= -\left(1+\frac{v_2}{v_2+0.5}\right)(0.5-b)\end{aligned}$$

$$\begin{aligned}\widehat{U}^{x_2}_{\{x_3,x_4,x_5\}} &= -\left(\frac{0.5}{v_1+v_2+0.5}(0.5-b)+\frac{v_2}{v_1+v_2+0.5}(1-2b)+\frac{v_1}{v_1+v_2+0.5}(1-a-b)\right)\\ &= -\left(\frac{2v_2+0.5}{v_1+v_2+0.5}(0.5-b)+\frac{v_1}{v_1+v_2+0.5}(1-2b+b-a)\right)\\ &= -\left(\frac{2v_1+2v_2+0.5}{v_1+v_2+0.5}(0.5-b)+\frac{v_1}{v_1+v_2+0.5}(b-a)\right).\end{aligned}$$

Therefore

$$\begin{aligned}-\left(\widehat{U}^{x_2}_{\{x_1,x_3\}}+\widehat{U}^{x_2}_{\{x_3,x_4,x_5\}}\right) &= \left(\frac{v_1}{v_1+0.5}+\frac{v_1}{v_1+v_2+0.5}\right)(b-a)\\ &\quad+\left(\frac{2v_1+2v_2+0.5}{v_1+v_2+0.5}+\frac{0.5}{v_1+0.5}\right)(0.5-b)\end{aligned}$$

We conclude

$$\begin{aligned}U^{\text{out}}_{x_2} &= -v_1\left(\frac{v_1}{v_1+0.5}+\frac{v_1}{1-v_1}\right)(b-a)-v_1\left(\frac{1.5-2v_1}{1-v_1}+\frac{0.5}{v_1+0.5}\right)(0.5-b)\\ &\quad-(1.5-4v_1)(0.5-b)\end{aligned}$$

The condition $U^{\text{in}}_{x_2}\geq U^{\text{out}}_{x_2}$ is

$$(0.5-b)+\frac{(1-2v_1)\,v_1}{v_1+0.5}(b-a)\leq v_1\left(\frac{v_1}{v_1+0.5}+\frac{v_1}{1-v_1}\right)(b-a)$$

$$+v_1\left(\frac{1.5-2v_1}{1-v_1}+\frac{0.5}{v_1+0.5}\right)(0.5-b)+(1.5-4v_1)(0.5-b)$$

which, after manipulations, reduces to

$$\frac{0.5-b}{b-a}\geq\frac{v_1(2-v_1)}{v_1-v_1^2+0.5}. \tag{14}$$

Thus, the parameters have to be such that inequality 14 is satisfied

B.1.3 No Exit for the Party Located at 0.5

In case of exit the votes are split equally between x_2 and x_4. They both end up with a share of seats $v_2+\frac{v_3}{2}$, while x_1 and x_5 get v_1 each. The policy outcomes are as follows:

- When x_1 and x_2 are formateur, which happens with probability 50%, the coalition $\{x_1, x_2, x_4\}$ is formed.
- When x_4 and x_5 are formateurs, which happens with probability 50%, the coalition $\{x_2, x_4, x_5\}$ is formed.

It is clear that x_3 is worse off leaving, since it is worse off for each possible coalition formed. We conclude that there are no incentives to exit for x_3.

B.2 No Incentives to Enter

We now show that there is no profitable entry. We will assume that δ is very low, so it is impossible to gain seats by entering at a position which is at a distance greater that 2ε from one of the peaks. So, meaningful entry will have to be close to the existing parties (at most 2ε away). Furthermore, given the symmetry of the density function, it is sufficient to check that there are no incentives to enter in the interval $[0, 0.5)$. The reasoning for citizens located in the interval $(0.5, 1]$ is symmetric.

B.2.1 No Profitable Entry for Citizens Located at $(a - 2\varepsilon, a)$

By entering at position $y \in (a - 2\varepsilon, a)$ a citizen gets a vote share

$$v_y = F\left(\frac{a+y}{3}\right) < \frac{1}{2}v_1$$

while the seats of all the other parties are unchanged. In that case.

- If the formateur is y or x_1 then the coalition $\{y, x_1, x_2, x_3\}$ is formed.
- If the formateur is x_2 then the coalition $\{x_1, x_2, x_3\}$ is formed.
- In all other cases the outcome is the same as in the case of no entry.

In this case the gains are when the formateur is in the set $\{y, x_1\}$ and the loss when the formateur is x_2, since in that case y is excluded from the governing coalition and the weight of x_1 on the policy choices is reduced from $\frac{v_1}{v_1+0.5}$ to $\frac{v_1-v_y}{v_1-v_y+0.5}$. For ε small enough the second effect is stronger than the first, so entry is not profitable.

B.2.2 No Profitable Entry for Citizens Located at $(a, a + 2\varepsilon)$

By entering at position $y \in (a, a + 2\varepsilon)$ a citizen gets a vote share

$$v_y = F\left(\frac{y+b}{2}\right) - F\left(\frac{a+y}{2}\right) < \frac{1}{2}v_1.$$

The seats are subtracted to party x_1, while the seats of all the other parties are unchanged. The outcome is as follows:

- If the formateur is x_1 or y then the coalition formed is $\{x_1, y, x_2, x_3\}$.
- If the formateur is x_2 then the coalition formed is $\{y, x_2, x_3\}$.
- In all other cases the outcome is the same as in the case of no entry.

The reasoning is the same as in the case of entry at $(a - 2\varepsilon, a)$.

B.2.3 No Profitable Entry for Citizens Located at $(b - 2\varepsilon, b)$

By entering at position $y \in (b - 2\varepsilon, b)$ a citizen gets a vote share

$$v_y = F\left(\frac{y+b}{2}\right) - F\left(\frac{a+y}{2}\right) < \frac{1}{2}v_2.$$

The seats are subtracted to party x_2, while the seats of all the other parties are unchanged. The outcome is as follows:

- If the formateur is x_1, y or x_2 then the coalition formed is $\{x_1, y, x_2, x_3\}$.
- If the formateur is x_3 then the coalition formed is $\{x_2, x_3, x_4\}$.
- In all other cases the outcome is the same as in the case of no entry.

In this case the gains are when the formateur is in the set $\{x_1, y\}$ and the loss when the formateur is x_3. Again, for ε small enough entry is not optimal.

B.2.4 No Profitable Entry for Citizens Located at $(b, b + 2\varepsilon)$

By entering at position $y \in (b, b + 2\varepsilon)$ a citizen gets a vote share

$$v_y = F\left(\frac{y+0.5}{2}\right) - F\left(\frac{b+y}{2}\right) < \frac{1}{2}v_2.$$

The seats are subtracted to party x_2, while the seats of all the other parties are unchanged. The outcome is as follows:

- If the formateur is x_1, x_2 or y then the coalition formed is $\{x_1, x_2, y, x_3\}$.
- If the formateur is x_3 then the coalition formed is $\{y, x_3, x_4\}$.
- In all other cases the outcome is the same as in the case of no entry.

In this case the gains are when the formateur is in the set $\{x_1, x_2, y\}$ and the loss when the formateur is x_3. Again, for ε small enough entry is not optimal.

B.2.5 No Profitable Entry for Citizens Located at $(0.5 - 2\varepsilon, 0.5)$

By entering at position $y \in (0.5 - 2\varepsilon, 0.5)$ a citizen gets a vote share

$$v_y = F\left(\frac{y + 0.5}{2}\right) - F\left(\frac{b + y}{2}\right) < \frac{1}{2}v_3.$$

The seats are subtracted to party x_2, while the seats of all the other parties are unchanged. The outcome is as follows:

- If the formateur is x_1 or x_2 then the coalition formed is $\{x_1, x_2, y, x_3\}$.
- If the formateur is y or x_3 then the coalition formed is $\{x_2, y, x_3, x_4\}$.
- If the formateur is x_4 or x_5 then the entrant will be excluded from the alliance and the coalition will be $\{x_3, x_4, x_5\}$ with a lower weight for x_3.

The entrant is better off when the formateur is $\{x_1, x_2, y, x_3\}$ and worse off when $\{x_4, x_5\}$. For ε small enough entry is not worth it.

B.2.6 Parameters Values Leading to Equilibrium

The parameter of the model must satisfy inequalities (12), (13) and 14), as well the condition $v_2 + v_3 = 0.5$. A possible configuration of the parameters such that the three inequalities are satisfied is $a = 0.02$, $b = 0.34$, $v_2 = 0.18$, $v_3 = 0.32$.

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