# The Allocation of a Prize

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## Abstract

Consider agents who undertake costly effort to produce stochastic outputs observable by a principal. The principal can award a prize deterministically to the agent with the highest output or to all of them with probabilities that are proportional to their outputs. We show that the deterministic prize elicits more (expected, total) output when agents' abilities are evenly matched, otherwise the proportional prize does better. Therefore if agents' characteristics are sufficiently diverse compared to the noise on output, and are not heavily correlated (e.g., because they are picked i.i.d.), then the proportional prize will elicit more output. We in fact show that this is the case when any Nash selection (under the proportional prize) is compared with any individually rational strategy selection (under the deterministic prize), provided agents know each others' characteristics (the complete information case). When there is incomplete information, the same conclusion holds (but now we must restrict to Nash selections for both prizes).

In the event that the principal knows the distribution of agents' characteristics, we also compute the optimal scheme for awarding the prize (among all schemes conceivable).

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# 1 Introduction

Consider a principal who has hired several agents to work for him. Each agent can undertake costly unobservable effort to produce a stochastic observable output that the principal values. The principal, in exchange, has a pot of gold that is valued by the agents. The question is: how should the principal award the gold in order to elicit maximal expected output from the agents? Should he give all the gold to the best performer? Or should he divide it into $k$ successively smaller parts and award these as $1^{st}, 2^{nd}, ..., k^{th}$ prizes to the agents based upon the rank-order of their outputs? Or is there something else the principal can do?

We propose the following simple scheme. Let the principal "market" the pot of gold to the agents, on the understanding that they must pay for it with the output they have produced. How the gold gets allocated is then left to market forces. Indeed, suppose that agents $1, ..., n$ have put up supplies of $x_1, ..., x_n$ units of output (perhaps all they have produced, if they do not value the output per se but only the gold it can buy); and that the principal has put up $y$ units of gold on the other side of the market. The only price $p$ (of the output, in terms of gold) which will "clear" the market is[1] $p = y/(x_1 + ... + x_n)$, and this is tantamount to handing out $y$ to the agents in proportion to the quantities they have supplied.

We compare the "marketed prize" to the single prize which, in turn, is often better than multiple a priori fixed prizes (see Remark 3 in Section 8). Our main result is that, on balance, *the marketed prize elicits more expected total output from the agents than the single prize*.

This result can be transported to an entirely different sceanrio in which the participants in a contest are to be handed out a coveted *indivisible* prize, based upon their performance. What is needed is that the performance be susceptible to quantification via "scores". There are many situations where this, in fact, is the case. For instance, a manager can consider total revenue earned as the yardstick whereby to award the badge of honor, or promotion to a higher echelon, to the best salesman of the year. In a race, the time taken to complete it comes naturally to mind. Sometimes scores are of a more subtle structure: in a gymnastics contest each member of a jury gives subjective scores to different aspects of performance which are then aggregated to come up with final scores. (The reader can think of other examples.) One upshot of assigning numerical scores, and perhaps the reason why they are so prevalent, is that they enable us to judge not only who beat whom, but by how much. Was the race keenly contested or one-sided? What was

[1] the total demand for gold is $px_1 + ... + px_n$ which must equal the supply y

the margin of victory? These are questions that are often not without meaning, and amenable to plausible answers, which are seen in the way scores get defined in practice.

The time-honored tradition has been to award the prize to the contestant with the highest score. We call this the "deterministic" scheme, though it is deterministic only in the scores, and not necessarily in the effort undertaken by the contestants, since scores may be a random function of effort. But, in principle, the prize could be given with different probabilities to the contestants based upon the scores that they achieve. This opens up a wide class of schemes (see Section 10) of which the deterministic scheme is just one instance. The "proportional" scheme, which we juxtapose to the deterministic, awards the prize to all the contestants with probabilities that are proportional to their scores. This is tantamount to putting up a bunch of "lottery tickets" at the market, which the contestants can then "buy" with their scores. The use of lotteries to award prizes is extremely widespread in practice (a Google search yielded 3,390,000 results) and has been discussed in the theoretical literature starting with Tullock (1975) in the context of lobbying (see Section 1.1). However, to the best of our knowledge, it has not been studied in the "moral hazard" context of our paper where only the stochastic outputs of agents are observed and their efforts are not.

The proportional scheme is our proxy for awarding the prize in a manner that is less drastic than deterministic and more commensurate with performance. In its neighborhood lie many other schemes which will inherit its properties. So, for our purposes, it does not matter precisely how scores are defined, so long as they remain "relatively bounded" across the different definitions. (See Remark 2 in Section 8.) Needless to say, if performances are incapable of being sensibly quantified by scores, and can only be ranked, then the proportional scheme has no meaning and only ordinal schemes make sense.

If the aim of the scheme is to "create competition" and to get the contestants to strive hard, then we argue that on balance the proportional scheme outperforms the deterministic. Of course, were the contest designer to have detailed knowledge of the distribution of the characteristics of the contestants (i.e., their ability, aversion to effort, valuation of the prize), then he could come up with a carefully tailored scheme which is optimal among all schemes conceivable. (In section 10 we carry out such an exercise.) But often such knowledge is not at hand. The purpose then is to design a *robust* scheme, which is based solely on observable outputs and yet does well over a wide range of possible distributions "for generations to come". Both the deterministic and the proportional schemes are robust but, as was said, the proportional scheme elicits more output.

The intuition for this is crystal clear and best brought out with two agents who have complete information about each other's characteristics. (We show, in section 11, that our results are not marred when there is incomplete information, i.e., each agent is informed only of his own characteristics and has a probability distribution over those of his rivals.) Suppose the two agents' abilities are sufficiently uneven. Then the weak agent will not be able to overtake the output produced by the strong, even if he works hard. Since work is costly, he will tend to slacken. This, in turn, will cause the strong to also slacken since he can continue to win the prize with good probability even at low effort levels. The upshot is an equilibrium at which effort and output are low. In contrast, the proportional prize generates better incentives. By working hard and producing more output, the weak agent is able to achieve a decent increment in his probability of winning a prize, regardless of the fact that his output always lags behind his rival's. Thus he is inspired to work and creates the competition which also spurs his rival to work, culminating in an equilibrium where effort and output are high. That an egalitarian scheme, which distributes rewards commensurate with output produced, will often generate better incentives than an elitist scheme in which the rewards are reserved for the top few – this, in our view, is a theme of wide-ranging application and runs like a leitmotif in the design of mechanisms in several different contexts (see, e.g., Dubey and Geanakoplos (2005), Dubey and Haimanko (2003), Dubey and Wu (2001) where this theme has been explicitly emphasized.)

On the other hand, when abilities are more evenly matched (think of athletic stars competing in the Olympics), the deterministic prize will clearly elicit more effort. For if both work, they come out with nearly equal probabilities of winning the prize under either scheme. But if anyone slackens, his probability drops abruptly to zero under the deterministic scheme, while it drops less under the proportional scheme. Thus there is more to lose by slackening when the deterministic prize is in use.

Now if agents' characteristics are picked at random from a sufficiently dispersed set $X$, the probability that they are unevenly matched is high, so that the proportional scheme outperforms the deterministic scheme on average[2]. This is certainly true if agents' characteristics are picked *independently* from $X$ as we often postulate for mathematical convenience. But in fact it remains true much more generally, indeed so long as their characteristics are not heavily correlated or, to put it more graphically, the distribution is not concentrated in a small neighbor-

[2]Even more: when the proportional prize beats the deterministic (which happens frequently) it is by a big margin; whereas when it loses, it is by a small margin.

hood of the "diagonal".

The details of our results are as follows. We shall couch them in terms of the indivisible prize rather than the divisible pot of gold, though the two are completely isomorphic.

Let $\pi_D$ and $\pi_P$ denote the deterministic and proportional prizes; and let $\chi$ denote the characteristics of the agents. In Sections 2 and 3 we describe the strategic games $\Gamma_{\pi_D}(\chi)$, $\Gamma_{\pi_P}(\chi)$ engendered by $\pi_D$, $\pi_P$ when agents have complete information, i.e., each agent knows not only his own characteristics but also those of his rivals. This seems a tenable hypothesis when they compete in close proximity with each other.

Fix a distribution $\xi$ of agents' characteristics, obtained by picking them i.i.d. from an underlying set $X$ that admits enough "diversity". Let $\Phi(\chi)$ be an arbitrary selection of individually rational (IR) strategies in $\Gamma_{\pi_D}(\chi)$ (for almost all $\chi$ w.r.t. $\xi$). Similarly let $\Psi$ be an arbitrary Nash Equilibrium (NE) strategy selection for $\Gamma_{\pi_P}$ (or, in fact, a "Weak Nash Equilibrium" (WNS) selection, which is a somewhat looser notion - see Section 3). We show in Section 7 that the (expected, total) output under $\Phi$ (as we vary $\chi$ according to $\xi$) corresponds to high effort only by an elite coterie $K$ of high-ability agents, which is independent of the value $v$ of the prize and whose average size is a small fraction of the total number $|N|$ of agents if the noise on output is not too big. In contrast, the output under $\Psi$ is of the order of min $\{v, N\}$, entailing work across the whole population (see Section 6), and thus generally much higher than that produced under $\Phi$ (see Section 8). It also follows from our analysis that, when $\pi_D$ is replaced by $\pi_P$, the vast majority of non-elite agents is made better off at the expense of the elite coterie, as of course is the principal who is able to elicit more output.

In Section 9 we show a "regime change" between proportional and deterministic prizes (in terms of their efficacy in eliciting output) as we vary the similarity between the agents. It fully clarifies our intuition that the deterministic prize does better when agents are evenly matched but worse when they are different. Then we turn to the question of an optimal scheme (among all conceivable schemes) when the principal knows the distribution of agents' characteristics. This is a somewhat subtle matter, as the reader can see from our analysis in Section 10. Finally, in Section 11, we show that our theme remains intact when there is incomplete information among the agents. (For the convenience of the reader, most of the proofs have been put into an Appendix.)

## 1.1 Related Literature

There is a rich literature on lobbying, where agents put up bids of money and are awarded the prize either via the proportional scheme or the deterministic scheme ( called often "lottery" or "all-pay auctions", respectively). See, e.g., Tullock (1975,1980), Hillman and Riley (1989), Ellingsen (1991), Rowley (1991,1993), Bay, Kovenock and de Vries (1993,1996), Che and Gale (1997,1998), Nti (1999), Fang (2002) and the references therein. In most of this literature agents are assumed to have complete information about each other, and in all of it there is no issue of "moral hazard", i.e., the bids submitted by the agents are perfectly observable.

The literature on tournaments is also vast and does often emphasize moral hazard, i.e., observable outputs depend stochastically on unobservable effort. However proportional prizes do not seem to have received attention there. For tournaments with a single prize, see Lazear and Rosen (1981), Green and Stokey (1983), Nalebuff and Stiglitz (1983), Rosen (1986). Subsequent writers have considered multiple prizes whose number and worth is fixed prior to the contest, and which are then awarded to the contestants based upon the rank-order of their performance (Glazer and Hassin (1988), Broecker (1990), Anton and Yao (1992), Clark and Riis (1998), Krishna and Morgan (1998), Bulow and Klemperer (1999), Barut and Kavenock (1998), Moldovanu and Sela (2001)).

In both strands of literature, the focus is on analyzing Nash Equilibria (NE) (which are often unique and susceptible of being described by explicit formulae, given the special structural assumptions of the models).

What is new in our approach is that we compare the proportional and deterministic prizes in the presence of moral hazard. Our setting is sufficiently general so as not to preclude multiple Nash equilibria and to render it difficult to write explicit formulae for them (e.g., we assume nothing about disutility and productivity other than the fact that they are monotonic in effort in the appropriate sense). Nevertheless we are able to show that the worst NE under the proportional prize elicits more output than the best NE under the deterministic prize. In fact, we show somewhat more than this, since our comparison is based on WNE and IR as explained before, which are looser notions than NE (indeed IR is something which any solution concept, based on an arbitrary mixture of cooperation and competition, would be expected to satisfy). To the extent that this constrains agents' behavior less, our comparison is that much stronger (more credible?). Of course, the price we pay for our generality is that we stop at this comparison, and unable to discern any finer structure in agents' behavior, which would come to the fore were one to confine attention to NE, especially in scenarios where they are unique (as happens in some of the structured examples we study).

# 2 The General Model

## 2.1 The Agents

Each agent in our model has access to a finite subset $E \subset [0,1]$ of (fractional) effort levels. We assume $0 \in E$ and $1 \in E$. These represent no effort and maximal effort respectively.

An agent may choose any effort $e \in E$. In doing so, he incurs disutility $\delta(e) \geq 0$ and produces stochastic output given by a non-negative random variable $\tau(e)$ with finite mean $\mu(e)$. (We allow for the possibility that the range of $\tau(e)$ is discrete, even finite.) Effort 0 incurs disutility $\delta(0) = 0$ and produces output $\tau(0) = 0$ with certainty: it is just a proxy for "not participating" in the game.

Agents are driven to work by the lure of an indivisible prize, which is handed out to them by a prinicpal. If an agent places valuation $v > 0$ on the prize, and is awarded it with probability $p$, this yields him expected utility $pv$. (See, however, Remark 2 in Section 8.)

The triple $(\delta, \tau, v)$ characterizes an agent. We make *throughout* the following monotonicity and boundedness assumptions on the space of possible characteristics $(\delta, \tau, v)$:

$\delta, \mu$ are weakly monotonic in $e$ and there exist universal positive constants $c, C, d, D$ such that

$$ce < \delta(e) < Ce \tag{1}$$

and

$$de < \mu(e) < De \tag{2}$$

for all $e \in E \setminus \{0\}$.
(Note that, on account of *weak* monotonicity, there is no loss of generality in supposing that all agents have the same set $E$ of effort levels. The case of an arbitrary allocation of subsets of $E$ across agents is automatically included, provided that 0 and 1 belong to each agent's set.)

## 2.2 The Principal

Suppose now that we have a finite set $N$ of agents with characteristics $(\delta^n, \tau^n, v^n)_{n\in N}$. The principal cannot observe these characteristics, or the effort levels $(e^n)_{n\in N}$ that the agents might have undertaken; all he can see are the realizations $t = (t^n)_{n\in N}$ of the random outputs $(\tau^n(e^n))_{n\in N}$. Thus his **allocation $\pi$ of the prize** is given by a function

$$\mathbb{R}^N_+ \xrightarrow{\pi} [0,1]^N$$

where the component $\pi^n(t)$, of the vector $\pi(t)$, denotes the probability with which $n \in N$ is allocated the prize.

The principal is risk-neutral and cares only about the expected total output produced by the agents. To this end he can devise different allocation schemes $\pi$. The full class $\Pi$ of such schemes will be considered later in section 10. For the present, we focus on two particular schemes. In both $\pi^n(t) = 0$ for all $n \in N$ if $t = 0$, otherwise agents would be rewarded for not participating in the game.

The first scheme is familiar from practice: the prize is shared equally among the winners $W(t) = \{k \in N : t^k = max\{t^n : n \in N\}\}$

**Deterministic Prize** ($\pi_D$):

$$\pi^n_D(t) = \begin{cases} \frac{1}{|W(t)|} \text{ if } n \in W(t) \text{ and } t \neq 0 \\ 0 \text{ otherwise} \end{cases}$$

(Note that $\pi_D$ is deterministic only in the outputs, not necessarily in the effort levels.)

The other scheme is analytically simple to work with and, to our way of thinking, not without intuitive appeal. It amounts to handing out “lottery tickets” for the prize to each agent, proportional to the output that he produces:

**Proportional Prize**($\pi_P$):

$$\pi^n_P(t) = \begin{cases} \frac{t^n}{\sum_{k\in N} t^k} \text{ if } t \neq 0 \\ 0 \text{ otherwise} \end{cases}$$

# 3 The Strategic Game

We suppose that, in addition to knowing $\pi$, the agents know each others' characteristics $(\delta^n, \tau^n, v^n)_{n\in N}$. This seems to be a tenable hypothesis if agents compete in close proximity with one another. (In Section 11, we consider the case when an agent knows his own characteristics but is unsure about those of his rivals.)

Given $(\delta^n, \tau^n, v^n)_{n\in N}$ a strategic game is induced among the agents by the principal's choice of an allocation scheme $\pi$. The set of pure strategies of each agent $n \in N$ is $E$. Any $N$-tuple of pure strategies $\mathbf{e} = (e^n)_{n\in N}$ gives rise to a random vector $(\tau^n(e^n))_{n\in N}$ of outputs. The expected value $p^k$ of $\pi^k((\tau^n(e^n))_{n\in N})$ represents the probability of $k$ winning the prize and we define $k$'s payoff to be

$$F^k(\mathbf{e}) = p^k v^k - \delta^k(e^k)$$

Denote by $\Gamma$ the mixed extension of this game; and by $\Sigma^k \approx \Sigma$ the set of (mixed) strategies of $k$ in $\Gamma$, i.e. $\Sigma^k$ is just the set $\Sigma$ of probability distributions on $E$. (Without confusion, $F^k(\sigma)$ will continue to denote $k$'s payoff, when the mixed strategy N-tuple $\sigma \in \prod_{n\in N} \Sigma^n \approx \Sigma^N$ is played in $\Gamma$.)

# 4 Solution Concepts

## 4.1 Fixed Games

First we recall three standard concepts. For any $\sigma \equiv (\sigma^n)_{n\in N} \in \Sigma^N$, denote $\sigma^{-n} \equiv \left(\sigma^k\right)_{k\in N\setminus\{n\}} \in \Sigma^{-n} \equiv \prod_{k\in N\setminus\{n\}} \Sigma^k$.

The choice $\sigma \in \Sigma$ is **individually rational** (IR) in $\Gamma$ if

$$F^n(\sigma) \geq \max_{u\in\Sigma^n} \min_{v\in\Sigma^{-n}} F^n(u,v)$$

for all $n \in N$.

The choice $\sigma \in \Sigma$ is a **Nash Equilibrium** (NE) of $\Gamma$ if

$$F^n(\sigma) = \max_{\tilde{\sigma}^n\in\Sigma^n} F^n(\sigma^{-n}, \tilde{\sigma}^n)$$

for all $n \in N$.

The choice $\sigma^n \in \Sigma^n$ is **strictly dominant** (SD) for $n$ in $\Gamma$ if

$$F^n(\sigma^n, v) > F^n(u, v)$$

for all $u \in \Sigma^n \setminus \{\sigma^n\}$ and all $v \in \Sigma^{-n}$.

Finally we introduce a weakening of the notion of NE which will be relevant for us. The idea is to restrict the set of unilateral deviations available to an agent $n$ by only allowing him to shift probabilities (to whatever extent he wishes) from his current strategy $\sigma^n$ onto maximal effort 1. More precisely, denote

$$\Sigma^n(\sigma^n) = \{\tilde{\sigma}^n \in \Sigma^n : \tilde{\sigma}^n(e) \leq \sigma^n(e) \text{ for all e} \in \text{E} \setminus \{1\}\}$$

Then we say that $\sigma \equiv (\sigma^n)_{n\in N}$ is a **weak Nash strategy-tuple** (WNS) if

$$F^n(\sigma) = \max_{\tilde{\sigma}^n \in \Sigma^n(\sigma^n)} F^n(\sigma^{-n}, \tilde{\sigma}^n)$$

for all $n \in N$. If the above holds with $\{\sigma^n, 1\}$ in place of $\Sigma^n(\sigma^n)$, we say that $\sigma$ is a **very weak Nash strategy-tuple** (VWNS). Here the agent $n$ is only permitted to shift all the probabilities from $\sigma^n$ abruptly onto 1. (Notice that maximal effort 1 is the anchor for both these notions. Indeed $\mathbf{1} \equiv \{1, ..., 1\}$ is always a WNS in any game and hence also a VWNS.)

Let us denote by IR(Γ), NE(Γ), SD(Γ), WNS(Γ), VWNS(Γ), the set of all strategies that are IR, NE, SD, WNS, VWNS in the game Γ. It is evident that

$$SD(\Gamma) \subset NE(\Gamma) \subset IR(\Gamma)$$

and that

$$NE(\Gamma) \subset WNS(\Gamma) \subset VWNS(\Gamma)$$

reflecting the progressively stringent requirements of the definition as we go from IR to NE to SD, or from VWNS to WNS to NE. (Note also that, obviously, SD(Γ)$\neq \emptyset$ implies SD(Γ) = NE(Γ) = a singleton set.)

## 4.2 Spaces of Games

Suppose characteristics $\chi \equiv (\delta^n, \tau^n, v^n)_{n\in N}$ are picked from $X \times ... \times X \equiv X^N$ according to some probability distribution $\xi$ on $X^N$. (Throughout, as was said, we assume that the underlying set $X$ satisfies (1) and (2)). Fix an allocation scheme $\pi$. Then any $\chi \in X^N$ induces a mixed-strategy game among the agents (as discussed in section 3), which we shall denote $\Gamma_\pi(\chi)$. We wish to extend our solution concepts to the space of games specified by $\xi$.

Our focus will be on what happens for *almost all* $\chi$ according to $\xi$ $(a.a.\chi(\xi))$, i.e., for all $\chi$ except perhaps for those in a set of $\xi$-measure zero.

Let $\sigma : X^N \to \Sigma^N$ be a strategy selection. We say that $\sigma$ is a $\xi$-$\Phi$-selection under $\pi$ (where $\Phi \equiv$ IR or NE or SD or WNS or VWNS) if, writing $\sigma_\chi$ for $\sigma(\chi)$, we have $\sigma_\chi \in \Phi(\Gamma_\pi(\chi))$ for $a.a.\chi(\xi)$.

# 5 Expected Output

Given a space of games $(X^N, \xi)$ what matters, from the principal's point of view, is the expected total output produced by $\sigma : X^N \to \Sigma^N$. Recalling that $\mu^n(e)$ is the mean of $\tau^n(e)$, we see that for any $\chi = (\delta^n, \tau^n, v^n)_{n\in N} \in X^N$, this output is given by

$$Exp_\sigma(\chi) \equiv \sum_{n\in N} \sum_{e\in E} \sigma^n_\chi(e)\mu^n(e) \tag{3}$$

and so, integrating over $X^N$ according to $\xi$, the expected total output on $X^N$ is

$$Exp_{\xi,\sigma} \equiv \int_{X^N} Exp_\sigma(x)d\xi(x) \tag{4}$$

# 6 Proportional Prize: Expected Output from Weak Nash Strategies

## 6.1 Prelude

It is clear a priori that, for any $\chi \in X^N$ and any scheme $\pi$, the total expected output in $\Gamma_\pi(\chi)$ cannot exceed $|N|D$ since no agent produces more than $D$ when he chooses maximal effort $e = 1$ (see (2)). Also[3], supposing $v^n = v$ for all $n \in N$, the total expected disutility incurred by the agents at any individually rational strategy selection cannot exceed $v$, otherwise some agent is incurring negative utility and would be better off not participating in the game. But then expected total output (see (1), (2)) is at most $Dv/c$. Thus, the most this output can be is "of the order of " $\min(v, |N|)$.

[3]Given $\chi = (\delta^n, \tau^n, v^n)_{n\in N}$, and a vector $\alpha \equiv (\alpha^n)_{n\in N} >> 0$ of positive scalars, let $\chi(\alpha) \equiv (\alpha^n\delta^n, \tau^n, \alpha^n v^n)$. Then the games $\Gamma_\pi(\chi)$ and $\Gamma_\pi(\chi(\alpha))$ are "strategically equivalent" and all our solution concepts remain the same for them. So w.l.o.g., scaling utilities appropriately, one could imagine $v^n = v$ for all $n \in N$.

This is the flavor of our estimate in Theorem 1 below, showing that the proportional prize elicits a "decent quantum" of output from the agents.

## 6.2 A Precise Estimate

For $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$ denote $\underline{v}(\chi)$ =min$\{v^n : n \in N\}$ and define

$$\underline{v} = ess\ inf_{\xi}(\underline{v}(\chi))$$

**Assumption AI**

$$\underline{v} > DC/d$$

We now show that Weak Nash Strategies (WNS) elicit a decent quantum of output under the proportional prize.

**Theorem 1**

Suppose Assumption AI holds. Denote $e_{\min}$ ≡min$\{e : e \in E \setminus \{0\}\}$. Let $\sigma$ be a $\xi$-WNS-selection under $\pi_P$. Then

$$Exp_{\xi,\sigma} \geq \frac{1}{2}\min\{|\mathrm{N}|\mathrm{de}_{\min}, \frac{\mathrm{d}\underline{\mathrm{v}}}{\mathrm{C}} - \mathrm{D}\}$$

where $Exp_{\xi,\sigma}$ is as defined in (4).

**Proof**: See the Appendix.

**Remark 1**: Somewhat more sharply

$$Exp_{\xi,\sigma} \geq \max_{0<\mathrm{p}<1} \min\{\mathrm{p}|\mathrm{N}|\mathrm{de}_{\min}, (1-\mathrm{p})(\frac{\mathrm{d}\underline{\mathrm{v}}}{\mathrm{C}} - \mathrm{D})\}$$

which is achieved at $p^* = B/(A+B)$ where $A = p|N|de_{min}$ and $B = (d\underline{v}/C) - D$

A variant of Theorem 1 for *Very* Weak Nash Strategies (VWNS) may also be of interest. Let assumption AI$'$ be the strengthening of AI obtained by substituting $2C$ for $C$. Then we have

**Theorem** $1'$

Suppose Assumption AI$'$ holds. Let $\sigma$ be a $\xi$-VWNS-selection under $\pi_P$. Then

$$Exp_{\xi,\sigma} \geq \frac{1}{4} min\{|N|de_{min}, \frac{d\underline{v}}{C} - 2D\}$$

where $Exp_{\xi,\sigma}$ is as defined in (4).

**Proof**: See the Appendix.

It might help to see what Theorem 1 implies when the number of players increases. The following immediate Corollary asserts that the expected total output, elicited via WNS-strategy selections by the proportional prize, grows as fast as the minimum value of the prize or the number of players, whichever is smaller (modulo the very minor requirement, given in Assumption AI, no one values the prize *too* low).

**Corollary**

Suppose the set of players is increasing, i.e., $|N| \to \infty$, and the corresponding spaces $(X^N, \xi_N)$ satisfy Assumption I with $\underline{v}(N)$ in place of $\underline{v}$ (and $\xi_N$ in place of $\xi$). For each $N$, let $\sigma_N$ be a $\xi_N$-WNS-selection under $\pi_P$. Then

$$Exp_{\xi_N,\sigma_N} \geq O(min\{|N|, \underline{v}(N)\})$$

**Proof**: Obvious.

## 6.3 Variations on the Theme

### 6.3.1 Highly Valued Prizes

In Theorem 1, the maximum value $\bar{v} = max\{v^n : n \in N\}$ of the prize is allowed to be quite small, and then $-$ as was already said $-$ it is not possible to get too many agents to put in significant work under any allocation scheme $\pi$, simply because the disutility incurred jointly by them cannot exceed $\bar{v}$. But the value of the prize lies in the eyes of its beholders. Since we are speculating about populations of agents with highly variable characteristics, who will compete under the scheme $\pi_P$ "for generations to come", we may imagine the scenario when all the agents are of a mind to place high valuations on the prize. Alternatively we can think of the scheme $\pi_P$ being used to disburse a vast number of different indivisible prizes to the same population of agents, and then focus on the case when the prize is such that it happens to be valued by everyone.

In either setting, the mathematical analysis is the same. For $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$ recall that $\underline{v}(\chi) = min\{v^n : n \in N\}$. We will show that, for sufficiently high values of $\underline{v}(\chi)$, maximal effort $\mathbf{1} \equiv (1, ..., 1)$ can be implemented in a progressively stronger manner : first as an NE, then as a unique WNS and finally as an "almost-SD" of the game $\Gamma_{\pi_P}(\chi)$. To set the stage for this, we need to put a constraint on the space of games. (Recall that $\mu^n(e)$ denotes the mean of $\tau^n(e)$.)

**Assumption AII**

There exist universal positive constants $\beta$ and $\Delta > 0$ such that for $a.a.\chi(\xi)$, if $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$, then

$$\mu^n(1) - \mu^n(e) > \Delta$$

for all $e \in E \setminus \{1\}$ and all $n \in N$; and

$$\tau^n(e) < \beta$$

for all $e \in E$ and all $n \in N$

**Theorem 2**

Suppose Assumption AII holds. Then there exist thresholds $v_*$ and $v^*$ such that for $a.a.\chi(\xi)$:

$$\mathbf{1} \text{ is an NE of } \Gamma_{\pi_P}(\chi) \tag{5}$$

whenever $\underline{v}(\chi) > v_*$; and

$$\mathbf{1} \text{ is the unique WNS, hence also the unique NE, of } \Gamma_{\pi_P}(\chi) \tag{6}$$

whenever $\underline{v}(\chi) > v^*$.

**Proof**: See the Appendix.

Clearly there is a threshold $\tilde{v}$ (between $v_*$ and $v^*$) above which $\mathbf{1}$ becomes the unique NE of $\Gamma_{\pi_P}(\chi)$. Moreover, there is another threshold above which it is possible to implement $\mathbf{1}$ almost as an SD. Fix $\varepsilon > 0$ as well as $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$.

We shall say that **1** is "**strictly dominant 'up to error $\varepsilon$**" in the game $\Gamma_{\pi_P}(\chi)$ if maximal effort is a strictly dominant strategy for each player, conditional on the fact that his rivals' total output is at least $\varepsilon$, i.e.,

$$F^n(1|A) > F^n(\sigma^n|A)$$

for all $n \in N$ and all $\sigma^n \in \Sigma^n \setminus \{1\}$ and all $T|N| \geq A > \varepsilon$, where

$$F^n(\sigma^n|A) \equiv \sum_{e \in E} \sigma^n(e)[Exp_\tau(\frac{\tau^n(e)}{\tau^n(e)+A})v^n - \delta^n(e)]$$

**Theorem** $2'$

Suppose Assumption AII holds. Then for any $\varepsilon > 0$, there exists $v^{**}(\varepsilon)$ such that for $a.a.\chi(\xi)$:

$$\mathbf{1} \text{ is strictly dominant up to error } \varepsilon \tag{7}$$

in the game $\Gamma_{\pi_P}(\chi)$, whenever $\underline{v}(\chi) > v^{**}(\varepsilon)$

**Proof**: See the Appendix.

# 7 Deterministic Prize: Expected Output from Individually Rational Strategies

The following "Key Lemma" provides the crucial insight as to why the deterministic prize $\pi_D$ elicits limited output. Indeed it shows that only the most productive agent, along with those who stand a chance of beating him, set the bound on the output at any individually rational strategy-tuple.

Fix $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$. Denote by $h$ an agent (the "hero") who has maximal mean output under effort level 1, i.e., for all $n \in N$,

$$\mu^h(1) \geq \mu^n(1)$$

(where, recall $\mu^n(e)$ is the mean of $\tau^n(e)$). Define $K(\chi)$ to be the set of agents whose outputs at effort 1 have a positive probability of exceeding that of $h$, i.e.,

$$K(\chi) = \{n \in N : Pr[\tau^n(1) \geq \tau^h(1)] > 0\}$$

We shall show that the output under deterministic prize is commensurate with $|K(\chi)|$. First we need

## Assumption AIII

There exists a universal constant $B$ such that for $a.a.\chi(\xi)$, if $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$, then for all $n, k \in N$

$$\frac{v^n}{v^k} < B$$

and, moreover, $\tau^n(\tilde{e}) \succeq \tau^n(e)$ whenever $\tilde{e} > e$, where "$\succeq$" denotes first order stochastic dominance[4].

## Key Lemma

Suppose Assumption AIII holds. Then for $a.a.\chi(\xi)$ we have

$$Exp_\sigma(\chi) \leq 2|K(\chi)|B^2CD/c$$

where $Exp_\sigma(\chi)$ is as defined in (3)

**Proof**: See the Appendix.

### 7.1 Estimation of the Average Value of $|K(\chi)|$ with i.i.d. Agents

A natural scenario is that agents' characteristics are drawn i.i.d. from a sufficiently "diverse" set. To this end, we introduce an additional constraint on our space of games.

[4]Recall: $\tau^n(\tilde{e}) \succeq \tau^n(e)$ if Prob$\{\tau^n(\tilde{e}) \geq z\} \geq$ Prob$\{\tau^n(e) \geq z\}$ for all z $\in$ Range $\tau^n(\tilde{e}) \cup$ Range $\tau^n(e)$

**Assumption AIV**

1. There exist $a > 0, b > 0, \varepsilon > 0$ such that, for $a.a.\chi(\xi)$, if $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$, then for all $n \in N$

   - $\mu^n(1) \subset [a,b] \subset \mathbb{R}_{++}$
   - Support $\tau^n(1) \subset [\mu^n(1) - \varepsilon, \mu^n(1) + \varepsilon]$

2. As we vary $\chi$ on $X^N$ according to $\xi, \mu^n(1)$ is i.i.d. across $n \in N$ with uniform[5]probability in $[a,b]$.

We can think of $\varepsilon$ as the size of the random noise on output, and then the "diversity" of agents' characteristics is reflected for us in the fact that $\tilde{\varepsilon} \equiv \varepsilon/(b - a)$ is small.

**Lemma 1**

Suppose Assumption AIV holds. Then the expected value of $|\kappa(\chi)|$ under $\xi$ is at most $1 + (|N|\varepsilon/(b-a))$.

**Proof**

See the Appendix.

## 7.2 Expected Output

We are ready to state the main conclusion of this section.

**Theorem 3**

Assume Assumptions AIII and AIV hold. Let $\sigma$ be a $\xi$-IR-selection on $X^N$. Then

$$Exp_{\xi,\sigma} \leq \frac{2B^2CD}{c}(1 + (\frac{|N|\varepsilon}{(b-a)}))$$

**Proof**

Immediate from the Key Lemma and Lemma 1.

[5]Any probability with a continuous density $f$ on [a,b] would do. We take the density to be a constant for ease of calculation.

# 8 Proportional Versus Deterministic Prizes

## 8.1 Expected Total Output

Theorems 1 and 3 enable an immediate comparison between the (expected total) outputs elicited from WNS, IR strategy selections by $\pi_P, \pi_D$ respectively. Fix, for example, all the parameters $c, C, d, D, b, B, a, \underline{v}$ of the model (such that $\underline{v} > DC/d$) and suppose that Assumptions AI, III, IV hold. There exists a threshold $\bar{\varepsilon}$ such that, if $\varepsilon < \bar{\varepsilon}$, then

$$Exp_{\xi,\sigma} > Exp_{\xi,\tilde{\sigma}}$$

for any $\xi$-WNS-selection $\sigma$ and $\xi$-IR-selection $\tilde{\sigma}$. This is so because the lower bound on output given by Theorem 1 is independent of the noise $\varepsilon$, while the upper bound given by Theorem 2 goes to 0 as $\varepsilon \to 0$.

To get a better feel, it might help to consider a numerical example. Let B=C=D=1, c=d=0.75, $[a,b] = [1,11]$, $|N| = 50$, $E = \{0, 0.5, 1\}$, $\underline{v} = 12$. In effect, there are two real effort levels (shirk=0.5, work=1), considerable diversity in characteristics (measured by $b-a$, $C/c$, $D/d$ and less than 4% noise $\equiv \varepsilon/(b-a)$). There are 50 agents but the prize is worth only 12 and cannot possibly incite more than $12.(4/3) = 16$ agents to put in effort 0.5 or more.

By Theorem 1, the output is bounded *below* by (1/2)min$\{|N|de_{min}, (d\underline{v}/C) - D\}$= (1/2)min$\{150/8, 9-1\} = 4$ at any WNS-selection under the proportional prize. On the other hand, by Theorem 2, the output is bounded *above* by $2B^2CD/c(1 + (|N|\tilde{\varepsilon}/(b-a))) < 4/3(1+50(0.04)) = 4$ at any IR selection under the deterministic prize.

## 8.2 Welfare

When the deterministic prize is used only the players in the elite coterie $K(\chi)$ (whose size is $1 + [|N|\varepsilon/(b-a)]$ on average) get the prize with significant probability under any IR strategy tuple. More precisely, the remaining players in $N \setminus K(\chi)$ get the prize with probablity at most $\underline{v}(\chi)B\sum_{k\in K(\chi)} \delta^k(1)$ (See the proof of the Key Lemma in the appendix for this estimate.)

If the proportional prize is used then, at any WNS strategy tuple, not only does the expected total output go up for the principal as we just saw, but each player in $N \setminus K(\chi)$ wins the prize with much greater probability than before (at least $de_{min}/|N|D \equiv O(1/|N|)$ each, provided $de_{min}\underline{v}(\chi)/|N|D > Ce_{min}$, i.e., provided

$\underline{v}(\chi) > C|N|D/d)$. Thus for $\underline{v}(\chi)$ large enough, all the players in $N \setminus K(\chi)$, who constituted the impoverished majority under the deterministic prize, suddenly find their prospects brighten and are able to become well off by working hard. The elite coterie $K(\chi)$, of course, loses its status : the probabilities of winning the coveted prize drops from $O(1/|K(\chi)|)$ to $O(1/|N|)$ for each of its members, though they still must work so as to not lag behind the others. In short, the egalitarian distribution engendered by the proportional prize inspires all agents to work hard and considerably raises total output.

The principal and the impoverished majority $N \setminus K(\chi)$ should both applaud when $\pi_P$ replaces $\pi_D$ ; or, rather, the principal can count on the unconditional support of the majority when he institutes $\pi_P$ instead of $\pi_D$, and need only worry about having to brook the displeasure of the tiny elite coterie $K(\chi)$.

## 8.3 Large N and i.i.d. Agents

If we let $|N|$ increase in the i.i.d. setting of this section, then the proportional prize will not only elicit more total output (compared to the deterministic) averaged across $\chi$, but will in fact also elicit more output for $\chi$ occuring with high probability. Precisely, there is a threshold $\varepsilon^*$ such that if $\varepsilon < \varepsilon^*$ then the following holds:

**Assertion**
For any $\delta > 0$, there exists $m(\delta)$ such that if $|N| > m(\delta)$ then

$$Exp_{\sigma}(\chi) > Exp_{\tilde{\sigma}}(\chi) \tag{8}$$

with probability at least $1-\delta$, where $\sigma(\chi)$ and $\tilde{\sigma}(\chi)$ are arbitrary WNS and IR strategy-tuples in $\Gamma_{\pi_P}(\chi)$ and $\Gamma_{\pi_D}(\chi)$ respectively. (Indeed, by lowering the threshold $\varepsilon^*$, we may even strengthen (8) to

$$Exp_{\sigma}(\chi) > MExp_{\tilde{\sigma}}(\chi)$$

for any $M > 0$.)

This is a straightforward consequence of the law of large numbers (viewed in conjunction with Theorems 1 and 3) as the reader may easily check.

**Remark 2 (Bounded Deviation)**. Suppose productivity functions $\tau^n$ are altered to $h^n \circ \tau^n$ for differentiable $h^n : \mathbb{R}_+ \to \mathbb{R}_+$ and that the derivative of $h^n$ is bounded below by $\gamma^{-1}$ and above by $\gamma$ for some positive constant $\gamma$ (independent of n). Then it is clear that our results in this section will continue to hold. The alteration can be absorbed by changing the lower and upper bounds in (2) from $d, D$ to $\gamma^{-1}d, \gamma D$.

In the same vein take any utility function for the prize which is of "bounded deviation" from expected utility. Precisely, an agent's utility from getting the prize with probability $p$ could be a function $f(p)$ for which there exists positive constants $\alpha$ and $v$ such that $\alpha^{-1}pv \leq f(p) \leq \alpha pv$. This also does not affect the tenor of our results. It can be absorbed by replacing $B$ in (6) by $\alpha^2 B$.

**Remark 3 (Multiple Prizes)**. One might wonder what happens when $l \leq |N|$ deterministic prizes are used instead of a single prize (i.e., the pot of gold is split a priori into $l$ successively smaller parts to be handed out to the agents with the top $l$ outputs; with some suitable postulate on how agents value fractions of the pot, e.g., linearly).

When $|N| = 2$ it is evident that using two prizes is wasteful since the loser will always get the second prize for free.

If there are $l << |N|$ prizes, then again the proportional prize will perform better. The reason is as follows. Assume everyone works hard. Define $l$ "heroes" by the top $l$ mean outputs (as in section 7); and then define the coterie $K$ to consist of those agents whose outputs have a positive probability of overtaking the weakest hero. Arguing as in the proof of the Key Lemma, the maximal effort in $K$ will effectively bound the total output at any IR strategy-tuple, regardless of the values of the $l$ prizes.

Furthermore, as in Section 7.2, the expected size of $K$ will be small. Thus the proportional prize will outperform $l$ deterministic prizes when $l << |N|$.

We plan to explore the case of general $l$ in future work.

# 9 Regime Change (Two Agents with Variable Noise in Output)

We devote this section to an example which brings out our central theme: if agents are "similar" then the deterministic prize elicits more output, otherwise that distinction goes to the proportional prize. To this theme, one may adduce one more observation: if agents are chosen "at random" from a "sufficiently diverse" set of characteristics, then the probability that they are similar is small. The upshot is that the proportional prize elicits more output on average, as our analysis has revealed.

To better illustrate our theme, it will help to suppress the random choice of agents' characteristics. Thus our example is going to be particularly simple. There are only two agents i.e., $N = \{1,2\}$ and only two effort levels (besides the "0" which is tantamount to not participating in the game), i.e., $E = \{0, 1/2, 1\}$. For simplicity fix $\delta^1(1/2) = \delta^2(1/2) = 0$ (which is just a proxy for a very small positive number) and $\delta^1(1) = \delta^2(1) = \delta > 0$. Fix also two numbers $0 < a < b$. We shall vary the productive abilities $\tau_\varepsilon^n$ of $n = 1,2$ with a parameter $\varepsilon$. For effort level $1/2$, both agents produce output uniformly in the interval $[0,\varepsilon]$. For effort level 1, agent 1 produces uniformly in $[a, a+\varepsilon]$ while agent 2 produces uniformly in $[b, b+\varepsilon]$. Since $a < b$, agent 1 is weaker than agent 2, and the "dissimilarity" between them can be expressed by $\Delta(\varepsilon) \equiv Prob\left\{\tau_\varepsilon^2(1) > \tau_\varepsilon^1(1)\right\}$. As $\varepsilon$ increases from 0 to $\infty$, $\Delta(\varepsilon)$ falls from 1 (complete disparity) to $1/2$ (complete similarity). We may think of the $\varepsilon$-spread a "noise" which, when large, overwhelms the intrinsic difference $b-a$ in the agents' abilities and makes them very similar.

Taking our cue from Theorem 2, our goal is to implement $\mathbf{1} = \{1,1\}$ as an NE. For simplicity we suppose $v^1 = v^2 = v$ and inquire about the values[6] $v$ of the prize for which $\pi = \pi_D$ or $\pi_P$ implements $\mathbf{1}$ as an NE given $\varepsilon$. Indeed, since we have fixed $\delta^1$ and $\delta^2$, and are going to deduce $v$, the only exogenous variable is $\varepsilon$ which defines the productivity functions $\tau_\varepsilon^1(e), \tau_\varepsilon^2(e)$. Thinking of $\chi \equiv (\delta^n, \tau_\varepsilon^n)_{n\in\{1,2\}}$ as the "*pre*characteristics" of the agents, the space from which $\chi$ is chosen will be taken to be of the form $X(\alpha,\beta) = \left\{\left(\tau_\varepsilon^1, \tau_\varepsilon^2\right) : \alpha \le \varepsilon \le \beta\right\}$. (Notice that the same noise $\varepsilon$ is used for each agent.)

For any given $\chi \equiv (\delta^n, \tau_\varepsilon^n)_{n\in\{1,2\}} \approx \varepsilon$ and $v^1 = v^2 = v$, we have the game $\Gamma_\pi(\varepsilon, v)$ where $\pi = \pi_D$ or $\pi_P$. A little reflection reveals that if $\mathbf{1}$ is an NE of $\Gamma_\pi(\varepsilon, v)$, then $\mathbf{1}$ is also an NE of $\Gamma_\pi(\varepsilon, \tilde{v})$ for all $\tilde{v} > v$. Thus we can measure the "efficacy" of $\pi$ by the smallest value $v(\pi,\varepsilon)$ of $v$ for which $\pi$ implements $\mathbf{1}$ as an NE, given precharacteristics $\varepsilon$. This is given by

$$v(\pi,\varepsilon) = inf\{v \in \mathbb{R}_+ : \mathbf{1} \in NE(\Gamma_\pi(\varepsilon, v))\}$$

First let us restrict to the situation when $\alpha = \beta$, so that $X(\alpha,\beta) \equiv X(\varepsilon)$ is a singleton. We shall show that there is a threshold $\varepsilon^*$ (which depends on a,b) such

[6] This is *not* to say that the principal can strategically vary the value $v$ of the prize $-$ that value is not his to vary; it lies in the eyes of the agents who behold the prize. *We*, the analysts, vary $v$ in order to pinpoint the population of agents (or, of prizes) for which a given $\pi$ implements $\mathbf{1}$ as an NE.

that a "regime change" occurs there:

$$v(\pi_P, \varepsilon) - v(\pi_D, \varepsilon) = \begin{cases} \text{+tive if } \varepsilon < \varepsilon^* \\ \text{-tive if } \varepsilon > \varepsilon^* \end{cases}$$

i.e. the proportional prize $\pi_P$ beats the deterministic prize $\pi_D$ when the agents are in $[0, \varepsilon^*)$, i.e., are sufficiently dissimilar, whereas it loses to $\pi_D$ when similarity sets in for $\varepsilon > \varepsilon^*$. In our example, for $a = 2$ and $b = 3$, $\varepsilon^* \approx 2.8$. Thus if one restricts noise so that the output of "shirk" ($e = 1/2$) cannot overtake the output produced by the strong agent ($n = 1$) when he "works" ($e = 1$), then we must have $\varepsilon < 3$, implying that $\pi_P$ beats $\pi_D$ with probability $2 \cdot 8/3 \approx 0.93$ (assuming all $\varepsilon$ in $[0, 3]$ to be equally likely); if the overtaking can occur with probability at most 0.2, then $\varepsilon - 3 < 0.2\varepsilon$, i.e., $\varepsilon < 3/.8$, in which case $\pi_P$ beats $\pi_D$ with probability $2.8/(3/.8) \approx 0.7$.

Let us verify the existence of the threshold $\varepsilon^*$. For the game on $(N, X(\varepsilon))$, let $\Delta\bar{\pi}_D^n(\varepsilon) =$ increase in probability of winning the prize for $n$, when he switches from effort $e = 1/2$ to $e = 1$ (assuming that his rival is at $e = 1$, and that the deterministic prize $\pi_D$ is being used). Similarly, define $\Delta\bar{\pi}_P^n$ for the proportional prize $\pi_P$. Then clearly

$$v(\pi_D, \varepsilon) = \frac{\delta}{\min\left\{\Delta\bar{\pi}_D^1(\varepsilon), \Delta\bar{\pi}_D^2(\varepsilon)\right\}}$$

and

$$v(\pi_P, \varepsilon) = \frac{\delta}{\min\left\{\Delta\bar{\pi}_P^1(\varepsilon), \Delta\bar{\pi}_P^2(\varepsilon)\right\}}$$

Denoting the two minima by $\min_D(\varepsilon)$ and $\min_P(\varepsilon)$ respectively, we see that

$$min_P(\varepsilon) > min_D(\varepsilon) \Longleftrightarrow \pi_P \text{ beats } \pi_D$$
$$min_D(\varepsilon) > min_P(\varepsilon) \Longleftrightarrow \pi_D \text{ beats } \pi_P$$

It is easy to compute all these terms for our simple example. Indeed

$$\Delta\bar{\pi}_D^1(\varepsilon) = \frac{(\max\{\varepsilon - b + a, 0\})^2 - (\max\{\varepsilon - b, 0\})^2}{2\varepsilon^2}$$
$$\Delta\bar{\pi}_D^2(\varepsilon) = 1 - \frac{(\max\{\varepsilon - b + a, 0\})^2 - (\max\{\varepsilon - a, 0\})^2}{2\varepsilon^2}$$

and

$$\Delta\bar{\pi}_P^1(\varepsilon) = \int_b^{b+\varepsilon}\left[\int_a^{a+\varepsilon}\frac{x}{x+y}dx - \int_0^{\varepsilon}\frac{x}{x+y}dx\right]dy$$
$$\Delta\bar{\pi}_P^2(\varepsilon) = \int_a^{a+\varepsilon}\left[\int_b^{b+\varepsilon}\frac{y}{x+y}dy - \int_0^{\varepsilon}\frac{y}{x+y}dy\right]dx$$

Taking $b = a+1$, and integrating by parts, yields

$$\Delta\,\bar{\pi}_P^1(\varepsilon) = F(\varepsilon,y) - F(0,y) + F(a,y) - F(a+\varepsilon,y)\Big|_{y=a+1}^{y=a+1+\varepsilon}$$

where $F(c,y) \equiv \frac{1}{2}\left(y^2-c^2\right)\ln(y+c) - \frac{1}{4}y^2 + \frac{1}{2}cy\ (= \int y\ln(c+y)dy)$. And $\Delta\bar{\pi}_P^2(\varepsilon)$ is an identical expression, obtained by swapping $a$ with $a+1$.

We may now (with the help of MAPLE, and taking $a=2$ and $b=3$) plot $\min_D(\varepsilon)$, $\min_P(\varepsilon)$ and $\min_P(\varepsilon) - \min_D(\varepsilon)$ against $\varepsilon$ in Figures 1, 2, 3 below. In Figure 3 we see that the threshold $\varepsilon^*$ is $\approx 2.8$.

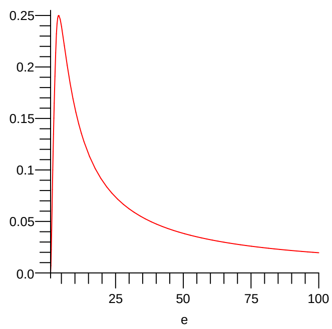


Figure 1: $min_D(\varepsilon)$

Turning to broader spaces $X(\alpha,\beta)$ with $\alpha < \beta$, first notice that $\Delta\bar{\pi}_D^1(\varepsilon) = 0$ if $\varepsilon \leq 1$ (for in this case agent 1 always produces below $b$, while agent 2 always produces above $b$ with effort level 1). Thus $v(N,\pi_D,X(\alpha,\beta)) = \infty$ if $\alpha < 1$. Since $\Delta\pi_P^n(\varepsilon) > 0$ for all $\varepsilon$ and $n \in \{1,2\}$, $v(N,\pi_P,X(\alpha,\beta)) < \infty$. It follows that $\pi_P$ is better than $\pi_D$ for all $(\alpha,\beta)$ if $\alpha < 1$. This is also true by our earlier discussion if $\beta < \varepsilon^* \approx 2.8$.

Figure 3 further reveals that when $\pi_P$ beats $\pi_D$, it does so most of the time by a large margin (e.g. by more than 0.1 for $0 < \varepsilon < 2$); whereas when it loses to $\pi_D$, the margin of loss is small ($\leq 0.1$).

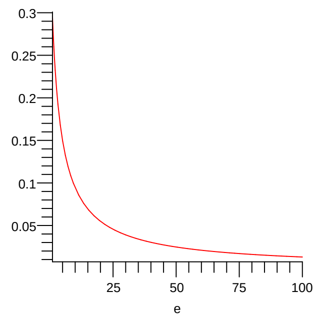


Figure 2: $min_P(\varepsilon)$

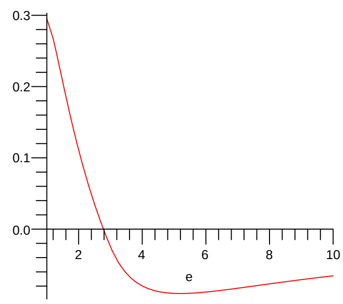


Figure 3: $min_D(\varepsilon) - min_P(\varepsilon)$

An alternative way in which to vary the productivities $\tau^1(1)$, $\tau^2(1)$ of agents $1,2$ is as follows. Fix $0 < a < b$. Let $\bar{N}(\sigma^2)$ be the "truncated" Normal distribution:

$$\bar{N}(\sigma^2) = \frac{N\left((a+b)/2, \sigma^2\right)}{\left(N\left((a+b)/2, \sigma^2\right)\right)[a,b]}$$

where the numerator is the standard normal distribution with mean $(a+b)/2$ and variance $\sigma^2$ and the denominator is the probability of the interval $[a,b]$ under that distribution. In short, $\bar{N}(\sigma^2)$ is the probability distribution induced by $N\left((a+b)/2, \sigma^2\right)$, conditional on being in $[a,b]$.

Pick $x$ i.i.d. according to $\bar{N}(\sigma^2)$ for each agent $n$, and let $\tau^n(1)$ be uniformly distributed in $(x-\varepsilon, x+\varepsilon)$ (where $\varepsilon$ is suitably small and fixed). As we increase $\sigma$ from 0 to $\infty$, the chances of "similarity" between the two agents fall from maximal to minimal. There will be a threshold $\sigma^*$ such that $\pi_P$ beats $\pi_D$ if, and only if, $\sigma > \sigma^*$ (according to the first criterion, or the third criterion, outlined in Section 6.1 with the threshold being higher, of course, for the first criterion). The verification, being straightforward, is omitted.

# 10 The Optimal Allocation of a Prize

We shall consider the class $\Pi$ of all allocation schemes $\pi : R_+^N \to [0,1]^N$ which satisfy the following conditions:

(i) (Scale Invariance)

$$\pi(rt) = \pi(t) \text{ for all scalars } r > 0$$

(ii) (Anonymity)
For any permutation $\omega : N \to N$,

$$\pi(\omega t) = \omega(\pi t)$$

(iii) (Monotonicity)

$$\pi^n(t) \geq \pi^k(t) \text{ whenever } t^n \geq t^k$$

(iv) (Disbursal)

$$\sum_{n \in N} \pi^n(t) = 1$$

if $t \neq 0$; and the sum is 0 if $t = 0$

The construction of an "optimal" scheme (defined below) in $\Pi$ for a given set $X$ of characteristics $(\delta^n, \tau^n)_{n \in N}$ is a delicate matter. We shall discuss it in the simple setting of two agents (i.e., $N = \{1,2\}$) with binary effort levels and deterministic[7] output. The effort levels are "shirk" ($e = 1/2$) and "work" ($e = 1$) — in addition, of course, to effort level 0 for not participating in the game. So $E = \{0, 1/2, 1\}$. The disutility of effort is fixed in $\chi$ (with[8] $\delta^n(1/2) = 0$ and $\delta^n(1) = \delta$

[7]Our analysis will not be disrupted by the introduction of small noise: the optimal $\pi^*$ will continue to be "approximately" optimal.

[8]We take $\delta^n(1/2) = 0$ for simplicity (recall that $\delta^n$ is permitted to be *weakly* increasing, as pointed out in footnote 1). But our analysis remains intact if $\delta^n(1)$ is sufficiently larger than $\delta^n(1/2) > 0$ (as can easily be checked.)

for $n = 1,2$). What varies is the productivity of an agent. Let $\tau(e,s)$ denote the deterministic output of each agent when he exerts effort $e \in \{1/2, 1\}$ and and is endowed with "skill" $s \in [0,K]$, so that we may take $X \approx [0,K] \times [0,K]$.

As in section 9, we shall take the implementability of maximal effort **1** as our criterion, and accordingly define

$$v(\pi, \chi) = inf\{v \in \mathbb{R}_+ : \mathbf{1} \in NE(\Gamma_\pi(\chi, v))\}$$

and

$$v(\pi) = sup\{v(\pi, \chi) : \chi \in X\}$$

Thus $v(\pi)$ is the smallest value $v = v^1 = v^2$ of the prize which implements **1** uniformly over $X$ when the scheme $\pi$ is used. We define $\hat{\pi}$ to be **optimal** in $\Pi$ for $X$ if

$$v(\hat{\pi}) \leq v(\pi)$$

for all $\pi \in \Pi$. Our goal in this section is to construct such an optimal scheme.
For brevity, denote $\tau(1/2,s) \equiv \tau(s)$ and $\tau(1,s) \equiv \tau^*(s)$. We make some natural monotonicity assumptions on $\tau$ and $\tau^*$, along with a form of "decreasing returns to skill":

**Assumption A V** Both $\tau : [0,K] \longrightarrow \mathbb{R}_+$ and $\tau^* : [0,K] \longrightarrow \mathbb{R}_+$ are strictly monotonic, with $\inf\{\tau^*(s) - \tau(s) : s \in [0,K]\} > 0$, and

$$\frac{\tau^*(s)}{\tau(s)} \leq \frac{\tau^*(s')}{\tau(s')} \qquad \text{whenever } s' < s$$

The displayed inequality says that the percentage gain in output, by switching from shirk to work, is a weakly decreasing function of the skill $s \in [0,K]$. It simplifies the analysis considerably. Indeed our goal is to incentivize an agent (of skill $s$) to switch from shirk to work, assuming his rival (of skill $t$) is working. The inequality above implies (see Lemma 2 in the Appendix) that our goal will be achieved for every $(s,t) \in [0,K] \times [0,K)$ if it is achieved for $(s,K)$ and $(K,s)$ for all $s \in [0,K]$; in other words, we need only to worry about incentivizing the agent in the following two extremal cases:

**Case A** His skill is $s \in [0,K]$ and his rival is working with skill $K$.
**Case B** His skill is $K$ and his rival is working with skill $s$.
Denote

$$R(s) = \frac{\tau^*(s)}{\tau^*(s)+\tau^*(K)}$$
$$r(s) = \frac{\tau(s)}{\tau(s)+\tau^*(K)}$$
$$\tilde{R}(s) = \frac{\tau^*(K)}{\tau^*(K)+\tau^*(s)}$$
$$\tilde{r}(s) = \frac{\tau(K)}{\tau(K)+\tau^*(s)}$$

When an agent switches from shirk to work, his fractional output goes up from

$$r(s) \text{ to } R(s) \text{ in Case A}$$
$$\tilde{r}(s) \text{ to } \tilde{R}(s) \text{ in Case B}$$

Denote $q(s) = 1 - \tilde{r}(s)$. It is clear from our assumptions that $q > R > r$ and that $R(s) = 1 - \tilde{R}(s), R(K) = \tilde{R}(K) = 1/2$

It will be useful for us to introduce one more function, which captures the simple form of $\pi \in \Pi$ when there are only two agents.

**Definition 1** A prize function is a weakly increasing function $p : [0,1] \to [0,1]$ satisfying

$$p(1-x) = 1 - p(x) \text{ for all } x.$$

The function $p$ is said to be *effective* at prize level $v$, if **1** = (work,work) is a Nash equilibrium for any pair $(s,t) \in [0,K] \times [0,K]$ of skills of the two players in the associated game.

Note that Assumption A II implies that, if $|N| = 2$ and $\pi \in \Pi$, then there exists a prize function $p$ such that $\pi^n(\tau^1, \tau^2) = p\left(\tau^n / \left(\tau^1 + \tau^2\right)\right)$, for $n \in N$, whenever $\tau^1 + \tau^2 \neq 0$ (justifying our name for $p$).

The following lemma will be useful.

**Lemma 3** The prize function $p$ is effective at level $v$ iff for all $s \in [0,K]$ we have

$$p(q(s)) - \delta/v \geq p(R(s)) \geq p(r(s)) + \delta/v$$

**Proof** As discussed earlier, $p(x)$ is effective iff for all $s \in [0,K]$

$$p\left(\tilde{R}(s)\right) \geq p(\tilde{r}(s)) + \delta/v \text{ and } p(R(s)) \geq p(r(s)) + \delta/v$$

Since $p\left(\tilde{R}(s)\right) = 1 - p(R(s))$, $p(\tilde{r}(s)) = 1 - p(q(s))$, the first inequality becomes

$$p(q(s)) - \delta/v \geq p(R(s))$$

which proves the result.

Define a sequence of points $0 = x_0, x_1, \ldots, x_l$ in $[0,1/2]$ by

$$x_i = \begin{cases} R(0) & \text{for} \quad i = 1 \\ \rho(x_{i-1}) & \text{for} \quad 1 < i \leq l \end{cases}$$

where

$$\rho(x) = \min\left(R\left(r^{-1}(x)\right), q\left(R^{-1}(x)\right)\right)$$

and $l$ is the smallest index $i$ for which $r^{-1}(x_i)$ is undefined. Note that since $q, R, r$ are all strictly increasing functions, so is $\rho$, and therefore $x_1, \ldots, x_l$ is an increasing sequence.

Now define $p^* : [0,1] \to [0,1]$ as follows:

$$p^*(x) = \begin{cases} i/2l & \text{for} \quad x_i \leq x < x_{i+1} \\ 1/2 & \text{for} \quad x_l \leq x \leq 1/2 \\ 1 - p^*(1-x) & \text{for} \quad 1/2 < x \leq 1 \end{cases}$$

**Theorem 3**

(i) Any effective scheme has prize level $\geq 2l\delta$.

(ii) $x \to p^*(x)\delta$ is an effective scheme with prize $2l\delta$.

**Proof** Let $p$ be an effective scheme with prize level $v$. Applying Lemma 3 with $s = 0$, we get

$$p(x_1) = p(R(0)) \geq p(r(0)) + \delta/v \geq \delta/v$$

Next let $s = r^{-1}(x)$ or $s = R^{-1}(x)$ according as $\rho(x) = R\left(r^{-1}(x)\right)$ or $q\left(R^{-1}(x)\right)$. Then by Lemma 3 we get

$$p(\rho(x)) \geq p(x) + \delta/v \text{ whenever } x, \rho(x) \in [0,1].$$

Applying this formula repeatedly we get

$$1/2 = p(x_l) \geq p(x_{l-1}) + \delta/v \geq \cdots \geq p(x_1) + (l-1)\,\delta/v \geq l\delta/v$$

which proves (i).

For (ii) we first show that, for any $s$, each of the two intervals $[r(s), R(s)]$ and $[R(s), q(s)]$ contains some "jump" point $x_i$. Indeed if $x = r(s)$ is in $[x_{i-1}, x_i)$, then $R(s) = R\left(r^{-1}(x)\right) \geq \rho(x) > \rho(x_{i-1}) = x_i$, hence $x_i \in [r(s), R(s)]$. The argument is similar for $[R(s), q(s)]$. Now by the definition of $p^*$ it follows that

$$p(q(s)) - 1/2l \geq p(R(s)) \geq p(r(s)) + 1/2l,$$

which is precisely the condition of Lemma 2 with $v = 2l\delta$.

# 11 Incomplete Information Game

Our main theme, namely that $\pi_P$ is better for the principal than $\pi_D$ when agents' characteristics are sufficiently diverse, has been established under the hypothesis that agents know each others' characteristics. Now we show that the theme remains intact even when information is coarsened in such a way that an agent is no longer sure of the characteristics of his rivals.

Let $E = \{0,1\}$ and $N = \{1,2\}$. Let $\delta^n(1) = 1$ and[9] $v^n = v > 1$ for $n = 1,2$; i.e., the uncertainty pertains only to the productivities $\tau^1$, $\tau^2$. Of course, $\tau_z^n(0) = 0$ as always, no matter what the "skill" $z$ of agent $n$ may be. Suppose that $\tau_z^n(1)$ is uniformly distributed on the interval $[z, z+\varepsilon]$, where $\varepsilon$ is a measure of the noise on the output. Furthermore suppose that the skills of the agents $n = 1,2$ are drawn independently from the intervals $[a_1, b_1]$ and $[a_2, b_2]$, with uniform probability (and that all this is common knowledge to the agents).

Since agent $n$ is informed of only his own skill, a strategy for him is given by a function

$$\sigma^n : [a_n, b_n] \to [0,1]$$

[9]If $v \leq 1$ then the only NE in $\Gamma^*_{\pi_D}$ or $\Gamma^*_{\pi_P}$ is that both agents never work (since effort 1 costs 1 which cannot be compensated by any probability of winning the prize)

where $\sigma^n(x)$ is the probability with which $n$ chooses effort 1 when his skill is $x$.

For any prize allocation scheme $\pi$, the game of incomplete information $\Gamma^*_\pi$ is then defined in the standard manner. (It depends not only on $\pi$ but also on the parameters $v, a_1, b_1, a_2, b_2, \varepsilon$ which we suppress because they will be understood. Our focus is on $\pi = \pi_P$ or $\pi_D$ which we keep track of in our notation.)

First consider the case when there is ex-ante symmetry between the agents and no noise

$$[a_1, b_1] = [a_2, b_2] = [0, 1] \text{ (say), and } \varepsilon = 0$$

Let $F^n_\pi((p, \sigma')|x)$ denote the payoff of $n$ in the game $\Gamma^*_\pi$, when he chooses effort 1 with probability $p$ and his skill level is $x$, while his rival chooses the strategy $\sigma'$. (Thus, if $n$'s strategy is $\sigma$, his payoff in $\Gamma^*_\pi$ will be $F^n_\pi(\sigma, \sigma') = \int_0^1 F^n_\pi((\sigma(x), \sigma')|x)dx$.) Notice that $F^n_\pi((1, \sigma')|x)$ increases[10] in $x$ (for fixed $n, \pi, \sigma'$), since $n$'s disutility of effort stays constant at 1 while his probability of winning the prize goes up[11]. Thus $n$'s best reply to $\sigma'$ is to switch from 0 to 1 at some "threshold" skill $c$, which solves $F^n_\pi((1, \sigma')|c) = 0$ i.e., denoting by $\sigma_c$ the strategy

$$\sigma_c(x) = \begin{cases} 1 \text{ if } x \geq c \\ 0 \text{ if } x < c \end{cases}$$

We see that $\sigma_c$ is a best reply to $\sigma'$ in the game $\Gamma^*_\pi$ if $F^n_\pi((1, \sigma')|c) = 0$. We conclude that $(\sigma_c, \sigma_c)$ is a[12] (symmetric) NE in $\Gamma^*_\pi$ if $F^n_\pi((1, \sigma_c)|c) = 0$. The unique $c(\pi)$ that solves this equation is computed rather easily for $\pi = \pi_P$ or $\pi_D$. Indeed we have,

$$F^n_{\pi_D}((1, \sigma_c)|c) = cv - 1$$

and

$$F^n_{\pi_P}((1, \sigma_c)|c) = cv + \int_c^1 (\frac{cv}{x + c})dx - 1$$

$$= cv[1 + ln\frac{1 + c}{2c}] - 1$$

[10]weakly in $\Gamma^*_{\pi_D}$ and strictly in $\Gamma^*_{\pi_P}$

[11]weakly in $\Gamma^*_{\pi_D}$ and strictly in $\Gamma^*_{\pi_P}$

[12]also "the", i.e., there is only one symmetric NE as the reader may easily verify.

which gives (denoting $c(\pi_D) \equiv c_D$ and $c(\pi_P) \equiv c_P$)

$$c_D = \frac{1}{v} \tag{9}$$

and

$$v = \frac{1}{c_P[1 + ln(\frac{1+c_P}{2c_P})]} \tag{10}$$

When $c_P = 0$, the right hand side of (10) is infinity by L'Hospital's rule while at $c = 1$, it is 1. Since $v > 1$ the solution of (10) is $c_P < 1$, hence we have $ln(\frac{1+c_P}{2c_P}) > 0$. Thus, for any $v > 1$, we deduce that $c_P > c_D$. In short, more player-types are working at NE under $\pi_P$ than under $\pi_D$ and hence $\pi_P$ elicits more expected output.

Now let noise increase (from 0 to infinity), still maintaining the ex-ante symmetry of the agents (i.e., $[a_n, b_n] = [0, 1]$ for $n = 1, 2$). Arguing as before, it is evident that threshold strategies will once again constitute NE. But for $\varepsilon$ large enough, the symmetry between agents will obtain even ex-post (to any desired level of accuracy) not just ex-ante, i.e., no matter what the realization of their respective skills, the two agents are nearly evenly matched since the large noise renders their skills irrelevant. In this event, as demonstrated in section 9, $\pi_D$ will elicit more effort than $\pi_P$. Indeed it is easy to verify (and we omit the routine algebra) that there exists an $\tilde{\varepsilon}$ such that

$$c_P(\varepsilon) < c_D(\varepsilon) \text{ if } \varepsilon < \tilde{\varepsilon}$$

and

$$c_P(\varepsilon) > c_D(\varepsilon) \text{ if } \varepsilon > \tilde{\varepsilon}$$

which asserts that, *unless the noise is so high as to make skills count for little $\pi_P$ outperforms $\pi_D$ in games of incomplete information* (exactly mirroring the situation of complete information (see section 9)).

Next let us consider the effect of allowing for ex-ante asymmetry of the incomplete information. To this end, let $[a_2,b_2]=[\Delta,1+\Delta]$ for $0<\Delta<1$[13] and $[a_1,b_1]=[0,1]$, i.e., agent 2's skills are $\Delta$-higher than 1's, so that $\Delta$ denotes the degree of asymmetry. For convenience, fix the noise $\varepsilon=0$. Arguing as in the ex-ante symmetric case (though, for more details, see Proposition 1 below), there again exist thresholds $c^n_D(\Delta), c^n_P(\Delta)$ such that $(\sigma^1_{c_D(\Delta)},\sigma^2_{c_D(\Delta)}),(\sigma^1_{c_P(\Delta)},\sigma^2_{c_P(\Delta)})$ constitute the symmetric NE of the games $\Gamma^*_{\pi_D},\Gamma^*_{\pi_P}$ respectively; and, moreover,

$$c^n_P(\Delta)<c^n_D(\Delta)$$

for $n=1,2$ and all $\Delta$ (unless $v$ is so small that no agent ever works in NE$-$ we implicitly eliminate such trivial NE by presuming $v$ is high enough). Thus $\pi_P$ always outperforms $\pi_D$ and, as anticipated, *the superiority of* $\pi_p$ *becomes more pronounced as the degree* $\Delta$ *of the asymmetry rises.*

The exact calculations for the asymmetric case emerge from the following proposition. Suppose an agent is informed that his rival's output is uniformly distributed in some interval $[z,z+\eta]\subset\mathbb{R}_+$ and that his own skill is $x$. Fix $x$ and think of $z,\eta$ as variable. We will compute two critical values $z_D\equiv z_D(x,\eta)$, $z_P\equiv z_P(x,\eta)$ such that the expected payoff of the agent is zero in $\Gamma^*_{\pi_D}$, $\Gamma^*_{\pi_P}$ if he chooses effort 1 and if $z=z_D$, $z=z_P$ respectively. Since this payoff varies inversely in $z$, the agent's best reponse to the rival is to choose effort 1 if $z<z_D$ and effort 0 if $z>z_D$ in the game $\Gamma_D$ (or, effort 1 if $z<z_P$ and 0 if $z>z_P$, in the game $\Gamma_P$). The critical values $z_D$, $z_P$ are given in proposition below.

**Proposition 1** *The critcal z-values are* $z_D=x-\eta/v$ *and* $z_P=\frac{\eta}{\exp(\eta/vx)-1}-x$ . *Moreover we have* $x(v-1)-\eta\le z_P\le x(v-1)$.

**Proof.** First consider $\pi_D$. Then $z=z_D$ implies $x=z+\eta/v$, and thus the player wins if the opponent's output lies in the interval $[z,z+\eta/v]$. This event has probability $(\eta/v)/\eta=1/v$ and gives expected payoff $v(1/v)-1=0$.

Now consider $\pi_P$. The expected payoff is

$$\frac{1}{\eta}\int_z^{z+\eta}\left(\frac{xv}{x+y}\right)dy-1=\frac{xv}{\eta}\ln\left(\frac{x+\eta+z}{x+z}\right)-1$$

[13]If $\Delta>1$ then we have the trivial situation that the highest skill-type of 1 cannot beat the lowest skill type of 2 which renders the deterministic prize ineffective, while the proportional still continues to elicit effort.

Setting this equal to zero and solving for $z$ we get

$$z = \frac{\eta}{\exp(\eta/xv) - 1} - x = z_P$$

For the bounds on $z_P$ we note that for an opponent of skill exactly $y^* = x(v-1)$ the payoff under $\pi_P$ is $\frac{xv}{x+y^*} - 1 = 0$. Thus if $z + \eta < y^*$ the payoff at each $y$ in $[z, z+\eta]$ is $\geq 0$, which implies $z_P \geq y^* - \eta$. Similarly if $z > y^*$, the payoffs in $[z, z+\eta]$ is $\leq 0$, which implies $z_P \leq y^*$. ■

We leave it to the reader to see how our results for the asymmetric case can be straightforwardly derived from this proposition. In fact, this proposition suffices also for the analysis of games of "partial information" which lie between what we, following others, have called games of "complete" and "incomplete" information. To be concrete suppose $[a_n, b_n]$ is partitioned into $k$ (say, equal) subintervals $[a_n + i\Delta, a_n + (i+1)\Delta]$ where $\Delta = (b_n - a_n)/k$ and $i = 0, 1, 2, ...k-1$. (When $k = 1$ we have "incomplete" information and as $k \to \infty$ we converge to "complete" information.) Each agent is now informed of his own exact skill and of the subinterval of $[a_n, b_n]$ in which his rival's skill lies. This defines a game of partial information in the obvious way (from his initial probability distribution on $[a_n, b_n]$, the agent can infer conditional probabilities of his rival's skill given the subinterval of $[a_n, b_n]$ in which it must lie).

We have not done the exact calculations, but it seems reasonably clear that $\pi_P$ outperforms $\pi_D$ for every $k$ not just for the two extreme points $k = \infty$ and $k = 1$ that have already been checked.

# Appendix

### Proof of Theorem 1

For brevity denote $Exp_\sigma$, defined in (3), as $Y$, i.e., $Y$ is the random variable which gives the expected total output of all the agents in $N$; and denote its expectation $Exp_{\xi,\sigma}$, defined in (4), as $\bar{Y}$. For $0 < p < 1$, consider the event

$$E = \{\chi \in X^N : Y(\chi) < \frac{\bar{Y}}{p}\}$$

It is evident that $\xi(E) \geq 1 - p$. Denote

$$F = \{\chi \in E : \exists k \in N s.t. \sigma_\chi^k(0) > 0\}$$

If $\xi(F) = 0$, every agent produces expected output at least $de_{min}$ almost everywhere in $E$, and so

$$\bar{Y} \geq (1-p)|N|de_{min} \tag{11}$$

If $\xi(F) > 0$, then there is an agent $n$ such that $\xi(F^n) > 0$ where

$$F^n = \{\chi \in E : \sigma^n_\chi(0) > 0\}$$

At each $\chi \in F^n$, let agent $n$ unilaterally change his strategy by shifting probability $\sigma^n_\chi(0)$ from effort 0 to effort 1. Since $n$ gets the prize with probability 0 when he chooses 0, and gets it (see (2)) with probability at least

$$\frac{d}{Y+D} \geq \frac{d}{(\bar{Y}/p)+D}$$

his gain in payoff is at least

$$\sigma^n(0)[\underline{v}(\frac{d}{(\bar{Y}/p)+D}) - C]$$

at every $\chi \in F^n$. Since $\sigma$ is a $\xi$-WNS-selection, we must have

$$\underline{v}(\frac{d}{(\bar{Y}/p)+D}) - C \leq 0$$

which gives

$$\bar{Y} \geq (1-p)(\frac{d\underline{v}}{C} - D) \tag{12}$$

Thus we have shown by (11) and (12) that

$$Exp_{\xi,\sigma} \equiv \bar{Y} \geq \min\{\mathrm{p|N|de_{min}}, (1-\mathrm{p})(\frac{\mathrm{d\underline{v}}}{\mathrm{C}} - \mathrm{D})\}$$

for all $0 < p < 1$. This establishes Remark 1 and, taking $p = 1/2$ also Theorem 1.

**Remark 3**: Observe that the above proof does not work if $\sigma$ is a $\xi$-VWNS-selection. For if the unilaterally deviating agent $n$ were to shift $\sigma^n_\chi(e)$ wholly onto 1 for *all* $e \in E \setminus \{1\}$, not just for $e = 0$, he may not stand to benefit because

- his increase in the probabilty of winning the prize when he switches from $e$ to 1, may be miniscual whenever $e \neq 0$ (becasue the probability was already close to 1 when he chose $e$), while the cost $\delta^n(1) - \delta^n(e)$ may be significant
- at the same time $\sigma^n(0)$ may be very small compared to $\sum_{e \neq 0,1} \sigma^n(e)$, so the gain in switching from 0 to 1 is outweighed by all the losses entailed in switching from $e \neq 0, 1$ to 1.

Thus in analyzing VWNS, we need to make sure that $\sigma^n(0)$ is big enough (we will ensure that it is at least 1/2 in the variation of the proof of Theorem 1 given below).

**Proof of Theorem** $1'$

Let $Y$ and $\bar{Y}$ be as in the proof of Thoerem 1. Consider the event

$$E = \{\chi \in X^N : Y(\chi) < 2\bar{Y}\}$$

It is evident that $\xi(E) \geq 1/2$. Denote

$$F = \{\chi \in E : \exists k \in N s.t. \sigma^k_\chi(0) > 1/2\}$$

If $\xi(F) = 0$, every agent produces expected output at least $(1/2)de_{min}$ almost everywhere in $E$, and so $\bar{Y} \geq (1/4)|N|de_{min}$, proving the theorem.

If $\xi(F) > 0$, then there is an agent $n$ such that $\xi(F^n) > 0$ where

$$F^n = \{\chi \in E : \sigma^n_\chi(0) > 1/2\}$$

At each $\chi \in F^n$, let agent $n$ unilaterally change his strategy from $\sigma^n_\chi$ to 1. Since $n$ gets the prize with probability 0 when he chooses 0, and gets it (see (2)) with probability at least

$$\frac{d}{Y+D} \geq \frac{d}{2\bar{Y}+D}$$

when he chooses 1, his gain in payoff is at least

$$\frac{1}{2} \cdot [\underline{v}(\frac{d}{2\bar{Y}+D})] - C$$

at every $\chi \in F^n$. Since $\sigma$ is a $\xi$-VWNS-selection, we must have

$$\underline{v}(\frac{d}{2\bar{Y}+D}) - 2C \leq 0$$

which gives

$$\bar{Y} \geq \frac{1}{4}(\frac{d\underline{v}}{C} - 2D) \tag{13}$$

proving the theorem.

## Proof of Theorem 2

First let us a note an obvious fact which we shall use repeatedly. Let $X$ be a nonnegative random variable, with upper bound $\tilde{B}$ and expectation $\tilde{M}$. Then, for any $\alpha \in (0,1)$ and $M \leq \tilde{M}$

$$Pr\{X > \alpha M\} > \frac{M - \alpha M}{\tilde{B} - \alpha M} \tag{14}$$

To see this, denote the LHS by $p$. Then $M \leq \tilde{M} \leq p\tilde{B} + (1-p)\alpha M$ which yields $p \geq (M - \alpha M)/(\tilde{B} - \alpha M)$.

We shall first establish (5) of Theorem 2. *Fix* throughout $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$ for which the bounds in Assumption AII apply (such $\chi$ occur with $\xi$-probability 1). For any $k \in N$, let $Y_{-k} \equiv \sum_{n \in N \setminus \{k\}} \tau^n(1)$ be the total output produced by the players in $N \setminus \{k\}$ when they all exert maximal effort. For brevity, denote $l \equiv |N| - 1 \equiv |N \setminus \{k\}|$. Then Exp $Y_{-k} \geq ld$ by (2). So, by Assumption AII and (14)(taking $M = ld, \tilde{B} = l\beta, \alpha = 1/2$ and noting that $\beta > d$) we obtain

$$Pr(Y_{-k} \geq ld/2 \geq \frac{ld/2}{l\beta - (ld/2)}) > \frac{d}{2\beta} \tag{15}$$

Given any realization $A > 0$ of total output $Y_{-k}$ produced by $N \setminus \{k\}$, let player $k$ unilaterally deviate from effort $e \in E \setminus \{1\}$ to 1. Then $k'$s probability of winning the prize goes up by (or, equivalently, others' probability of winning the prize goes down by)

$$
\begin{aligned}
& Exp_\tau[\frac{A}{A+\tau^k(e)} - \frac{A}{A+\tau^k(1)}] \\
&= Exp_\tau[\frac{A(\tau^k(1)-\tau^k(e))}{(A+\tau^k(e))(A+\tau^k(1))}] \\
&\geq \frac{AExp_\tau(\tau^k(1)-\tau^k(e))}{|N|^2\beta^2} \\
&= \frac{A(\mu^k(1)-\mu^k(e))}{|N|^2\beta^2} \\
&\geq \frac{A\Delta}{|N|^2\beta^2} \qquad (16)
\end{aligned}
$$

(The inequalities here follow from Assumption AII.) But $A \geq ld/2$ with probability at least $d/\beta$ by (15). Thus $k'$s gain in payoff, when he unilaterally deviates from $e \in E \setminus \{1\}$ to 1, is at least

$$\frac{d}{2\beta} \cdot \frac{ld}{2} \cdot \frac{\Delta}{|N|^2\beta^2} v^k \equiv Zv^k(\text{say})$$

On the other hand, his loss in payoff is at most $\delta^k(1) - \delta^k(e) \leq C$. Thus, if we choose $v_* > C/Z$, the gain outweighs the loss and we conclude that $\mathbf{1}$ is an NE of $\Gamma_{\pi_P}(\chi)$, proving (5). (Notice that, since $l \equiv |N|-1$, we have $Z \approx 1/(|N|)$ which implies $v_* = O(|N|)$ as expected from Theorem 1 according to which the total expected output is $O(min(|N|, v_*))$.)

We now turn to the proof of (6). First let us establish that there exists $v^+$ such that, if $\min\{v^n : n \in N\} > v^+$, then at any NE $\sigma$ of $\Gamma_{\pi_P}(\chi)$ we have

$$Exp_\tau\, Y_{-k} \geq ld/4 \qquad (17)$$

for all $k \in N$. Suppose provisionally that (17) is false, i.e., $Exp_\tau\, Y_{-\bar{k}} < ld/4$ for some $\bar{k} \in N$. Then

$$Pr(Y_{-\bar{k}} < ld/2) > 1/2 \qquad (18)$$

Clearly there exists $n \in N \setminus \{\bar{k}\}$ such that $\sigma^n(0) > 0$ (otherwise $Exp_\tau\, Y_{-\bar{k}} \geq ld$ contradicting our provisional hypothesis that $Exp_\tau\, Y_{-\bar{k}} < ld/4$.)

Let $n$ shift probability $\sigma^n(0)$ from 0 to 1. His loss in utility, from the extra work is at most $\sigma^n(0)C$. On the other hand, from (18) and Assumption AII, we see that his probability of winning the prize goes up by at least

$$\sigma^n(0).[\frac{\Delta}{(ld/2)+\beta}].\frac{1}{2}$$

We choose $v^+$ to ensure that the gain outweighs the loss i.e.,

$$v^+.[\frac{\Delta}{(ld/2)+\beta}].\frac{1}{2} > C$$

contradicting that $\sigma$ is a WNS of $\Gamma_{\pi_P}(\chi)$, and thus contradicting also (18), and thereby establishing (17)

Now by (14) and (17) (taking $M = ld/4, \alpha = 1/2, \tilde{B} = \beta l$ and noting that $\beta > d$ we derive

$$Pr(Y_{-k} > ld/8) \geq \frac{ld/8}{l\beta - (ld/8)} > \frac{d}{8\beta} \tag{19}$$

Consider any $k \in N$ and $e \in E \setminus \{1\}$. We shall show there exists $v^*$ such that, if $v^k > v^*$, then $k$ can improve his payoff by deviating from $e$ to 1 (assuming of course that all the other players are producing some given amount $\tilde{A} > ld/8$). Indeed, in view of (19) and (16) (using now $ld/8$ as the lower bound for $A$ in (16)), $k'$s gain in payoff is at least

$$\frac{d}{8\beta}.\frac{ld}{8}.\frac{\Delta}{|N|^2\beta^2}.v^k \equiv \tilde{Z}v^k(say)$$

while his loss is at most $C$. Thus it suffices to choose $v^* > C/\tilde{Z}$. Since $\tilde{Z} > Z$, we have $v^* > v^+$, proving (6).

## Proof of Key Lemma

Since $\chi \equiv (\delta^n, \tau^n, v^n)_{n\in N}$ is fixed, we shall suppress it and write $K \equiv K(\chi)$. Imagine the scenario when every agent in $K$ chooses 1. This defines probabilities $\pi_*^k > 0$ for $k \in K$ to win the prize.

It is evident that $(i) \sum_{k\in K} \pi_*^k = 1$ and $(ii)\, \pi_*^k$ is independent of the mixed strategies chosen by the players in $N \setminus K$ (each of whom gets the prize with zero probability in our scenario, since he is beaten for sure by the hero $h$). Furthermore, by Assumption $A\ III(2)$, $\bar{\pi}_*^k$ can only increase if any subset of players in $K \setminus \{k\}$ change to strategies other than 1. Hence we deduce that every player $k \in K$ can *guarantee* himself the payoff

$$\pi_*^k v^k - \delta^k(1)$$

by playing 1. Thus, if $\sigma \in IR(\Gamma_{\pi_D}(\chi))$,

$$F^k(\sigma) \geq \pi_*^k v^k - \delta^k(1)$$

for all $k \in K$. But clearly $F^k(\sigma) \leq \bar{\pi}^k(\sigma)v^k$ (denoting $\bar{\pi}^k(\sigma) \equiv k$'s probability of winning the prize when $\sigma$ is played), so we have

$$\bar{\pi}^k(\sigma) \geq \pi_*^k - \frac{\delta^k(1)}{v^k}$$

for all $k \in K$, which implies

$$\begin{aligned}\sum_{k \in K} \bar{\pi}^k(\sigma) &\geq \sum_{k \in K} \pi_*^k - \sum_{k \in K} \frac{\delta^k(1)}{v^k} \\ &= 1 - \sum_{k \in K} \frac{\delta^k(1)}{v^k}\end{aligned}$$

But then, putting $v \equiv v^1$ and observing $B^{-1}v \leq v^n \leq Bv$ for all $n \in N$, we have

$$\begin{aligned}\sum_{n \in N \setminus K} \bar{\pi}^n(\sigma) &= 1 - \sum_{k \in K} \bar{\pi}^k(\sigma) \\ &\leq \sum_{k \in K} \frac{\delta^k(1)}{v^k} \\ &\leq \frac{B}{v} \sum_{k \in K} \delta_k(1)\end{aligned}$$

So we obtain

$$\begin{aligned}\sum_{n \in N \setminus K} F^n(\sigma) &= \sum_{n \in N \setminus K} \left[ \bar{\pi}^n(\sigma)v^n - \sum_{e \in E} \sigma^n(e)\delta^n(e) \right] \\ &\leq Bv \sum_{n \in N \setminus K} \bar{\pi}^n(\sigma) - \sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e)\delta^n(e) \\ &\leq B^2 \sum_{k \in K} \delta^k(1) - \sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e)\delta^n(e)\end{aligned}$$

But each $n \in N \setminus K$ can guarantee a payoff of at least 0 by choosing effort level 0, so $F^n(\sigma) \geq 0$ since $\sigma \in IR(\Gamma_{\pi_D}(\chi))$, and so:

$$\sum_{n \in N \setminus K} F^n(\sigma) \geq 0$$

Combining the above two inequalities, we have

$$\sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e)\delta^n(e) \leq B^2 \sum_{k \in K} \delta^k(1)$$

Since $\delta^k(1) \leq C$ and $\delta^n(e) \geq ce$ by Assumption AI, we get

$$\sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e)e \leq B^2|K|\frac{C}{c}$$

Recalling also that $\mu^n(e) \leq Te$ by Assumption A I, we obtain

$$\sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e)\mu^n(e) \leq B^2|K|\frac{C}{c}T$$

Clearly, by our definition of $h$ and Assumption AI,

$$\sum_{k \in K} \sum_{e \in E} \sigma^n(e)\mu^k(e) \leq B^2|K|\mu^h(1)$$

$$\leq B^2|K|\frac{C}{c}T$$

(using the fact that $C > c$ in the last inequality). The above two inequalities prove the Key Lemma.

## Proof of Lemma 1

Let $0 < \varepsilon < 1$ be fixed.

For any $n$-tuple of real numbers $x = (x_1, \ldots, x_n)$, we write $M = \max(x_i)$ and define $N_\varepsilon$ to be the number
of $x_i$ in the open interval $(M - \varepsilon, M)$.

**Claim 2** Suppose the $x_i$ are independent and uniformly distributed in the closed interval $[0,1]$. Then $N_\varepsilon$ has the distribution $\min(n - 1, B(n,\varepsilon))$, where $B(n,\varepsilon)$ is the binomial distribution.

**Proof** For each $k \leq n-1$, we calculate the probability $\Pr(N_\varepsilon = k)$.

First suppose that $k < n-1$, and let $E_k$ denote the event that

$$\{x_1 \text{ is largest}\} \vee \{x_2, \ldots, x_{k+1} \in (x_1 - \varepsilon, x_1)\} \vee \{x_{k+2}, \ldots, x_n \in [0, x_1 - \varepsilon]\}$$

For $x$ in $[0,1]$ the density $\Pr(E_k | x_1 = x)$ is

| $x \leq \varepsilon$ | $x > \varepsilon$ |
|---|---|
| 0 | $\varepsilon^k (x-\varepsilon)^{n-k-1} dx$ |

.

Integrating over $x$ we get $\Pr(E_k) = \frac{\varepsilon^k (1-\varepsilon)^{n-k}}{n-k}$. Considering the possible permutations of the $x_i$ we get

$$\Pr(N_\varepsilon = k) = n \binom{n-1}{k} \Pr(E_k) = \binom{n}{k} \varepsilon^k (1-\varepsilon)^{n-k} \text{ for } n < k-1.$$

However for $k = n-1$ we get

$$\begin{aligned}
\Pr(N_\varepsilon = n-1) &= \Pr(N_\varepsilon = n-1 | \max(x_i) > \varepsilon) + \Pr(N_\varepsilon = n-1 | \max(x_i) \leq \varepsilon) \\
&= \binom{n}{n-1} \varepsilon^{n-1} (1-\varepsilon) + \varepsilon^n
\end{aligned}$$

and the result follows.

**Corollary** The expected value of $N_\varepsilon$ is $E(N_\varepsilon) = n\varepsilon - \varepsilon^n$.

**Proof** We calculate as follows

$$\begin{aligned}
E(N_\varepsilon) &= \sum_{k=0}^{n-2} k \binom{n}{k} \varepsilon^k (1-\varepsilon)^{n-k} + (n-1) \left[ \binom{n}{n-1} \varepsilon^{n-1} (1-\varepsilon) + \varepsilon^n \right] \\
&= \sum_{k=0}^{n} k \binom{n}{k} \varepsilon^k (1-\varepsilon)^{n-k} - \varepsilon^n = n\varepsilon - \varepsilon^n.
\end{aligned}$$

This completes the proof of Lemma 1.

# Lemma 2

Let $s \in (0,K)$ and $t \in (0,K)$, Then there exist $s' \in [0,K]$ and $t' \in [0,K]$ such that

(a) either $s' = K$ or $t' = K$

and

(b)
$$\frac{\tau^*(s')}{\tau^*(s') + \tau^*(t')} - \frac{\tau(s')}{\tau(s') + \tau^*(t')} \leq \frac{\tau^*(s)}{\tau^*(s) + \tau^*(t)} - \frac{\tau(s)}{\tau(s) + \tau^*(t)}$$

**Proof** Since $\tau^*$ and $\tau$ are strictly monotonic, there exist $\Delta > 0$ and $\Delta' > 0$ such that

$$s' \equiv s + \Delta \in [0,K], \qquad t' \equiv t + \Delta' \in [0,K] \tag{20}$$

and
$$\frac{\tau(s')}{\tau(s') + \tau^*(t')} = \frac{\tau(s)}{\tau(s) + \tau^*(t)} \tag{21}$$

Hence there exists a *maximal* pair $\Delta, \Delta'$ satisfying (20) and (21), and then either $s' = K$ or $t' = K$ (otherwise both $\Delta$ and $\Delta'$ could be increased slightly, still maintaining (20) and (21), and contradicting the maximality of $\Delta$, $\Delta'$).

In view of (21), to prove (b) it suffices to show that

$$\frac{\tau^*(s')}{\tau^*(s') + \tau^*(t')} \leq \frac{\tau^*(s)}{\tau^*(s) + \tau^*(t)} \tag{22}$$

which is equivalent to

$$\frac{\tau^*(t')}{\tau^*(s')} \geq \frac{\tau^*(t)}{\tau^*(s)} \tag{23}$$

as can be seen by dividing the numerator and the denominator of the LHS, RHS of (22) by $\tau^*(s'), \tau^*(s)$ respectively.

But a similar manuever shows that (21) is equivalent to

$$\frac{\tau^*(t')}{\tau(s')} = \frac{\tau^*(t)}{\tau(s)} \tag{24}$$

And, since $s' > s$, Assumption A V implies

$$\frac{\tau^*(s')}{\tau^*(s)} \leq \frac{\tau(s')}{\tau(s)} \tag{25}$$

From (24) and (25), we get

$$\frac{\tau^*(s')}{\tau^*(s)} \leq \frac{\tau(s')}{\tau(s)} = \frac{\tau^*(t')}{\tau(t)} \tag{26}$$

establishing (23), and thereby (22)

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