Quantitative Finance Area Exam

Stony Brook University
Applied Mathematics and Statistics

January 28th 2015

INSTRUCTIONS

You have 4 hours to do this exam.

Reminder: This exam is closed notes and closed books. No electronic devices are permitted. Computers, cell phones, tablets must all be turned off and put away for the duration of the exam. All problems are weighted equally.

PART 1: Do 2 out of problems 1, 2, 3.

PART 2: Do 2 out of problems 4, 5, 6.

PART 3: Do 2 out of problems 7, 8, 9.

PART 4: Do 2 out of problems 10, 11, 12.

For each problem an extra page is provided which you may use for scratch work or to continue your solution. On this cover page write which eight problems you want graded.

Problems to be graded:

Academic integrity is expected of all students at all times,
whether in the presence or absence of members of the faculty.
Understanding this, I declare that I shall not give, use,
or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number:

Signature:

Problem 1

Consider a custom European option on an underlying security with price $S(t)$. The pay-off function for the option at expiry T is

$$F(S(T), T) = \max[0, \min[3K - S(T), S(T) - K]].$$

- Sketch out the pay-off function at expiry. Clearly show the values $\{S(T), F(T)\}$ of the pay-off function at key inflection points.
- Write out a pricing formula for the price of the custom option at time t, $F(S(t), t)$, in terms of vanilla European puts and calls on S and, if necessary, any cash position saved or borrowed. Assume continuous compounding with a risk-free rate of return r_f. Use $P(S(t), K)$ and $C(S(t), K)$ to denote the price of a vanilla European put and call on S, respectively, at time t with strike price K.

Extra Paper 1

Problem 2

A security's price is found to be governed by the following mean reverting Itô process:

$$dS(t) = \eta(\theta - S(t))\,dt + \sigma\,dW(t)$$

Let $V(S(t), t)$ be a function which is at least twice differentiable by S and at least once differentiable by t. Find the Itô process which governs V.

Extra Paper 2

Problem 3

Consider a market composed of securities other than cash whose returns are governed by a mean vector $\boldsymbol{\mu}$, and covariance matrix $\boldsymbol{\Sigma}$. The market also has no transaction costs, no liquidity constraints, and no non-public information among investors. Further assume that all investors can invest or borrow at the risk-free rate r_f and that all investors are mean-variance optimizers.

Calculate an analytical expression for $\mathbf{x}$, the proportional (*i.e.*, $\mathbf{1}^T\mathbf{x} = 1$) allocation of securities in the market portfolio.

Extra Paper 3

Problem 4

1. Draw a picture of the efficient frontiers with/without the risk-free asset in the same risk-return plane. In the picture, you need to mark (1) the risk-free return rate, (2) the market portfolio and (3) explain what's the Sharpe ratio of the market portfolio.

2. Consider a two asset market where the mean return $\mu \sim \mathcal{N}(\bar{\mu}, V)$ with

$$\bar{\mu} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \quad V = \begin{bmatrix} 0.25 & 0.00 \\ 0.00 & 0.50 \end{bmatrix}$$

   Suppose we have the following view on this market

$$\mu_1 - 0.5\mu_2 = 0.04 + \epsilon,$$

   or equivalently,

$$\begin{bmatrix} 1 & -0.5 \end{bmatrix} \mu = 0.04 + \epsilon,$$

   where $\epsilon \sim \mathcal{N}(0, 0.2)$.

   Write the simplest possible expressions for the posterior mean vector $\bar{\mu}'$ and covariance matrix Γ and the MATLAB code to compute the values.

Extra Paper 4

Problem 5 Consider a market with n risky assets. Let $\tilde{r}_i$ denote the random return on asset i. Suppose the vector of random return $\tilde{r} \sim \mathcal{N}(\mu, \mathbf{V})$. The market has a risk-free asset with rate of return r_f.

Consider the following utility maximization problem:

$$\begin{array}{rl} \max & \mu' x + r_f x_0 - \frac{\lambda}{2} x' \mathbf{V} x, \\ \text{s.t.} & \mathbf{1}'\mathbf{x} + x_0 = 1. \end{array}$$

1. Solve for the optimal solution x^* when

$$n = 2, \quad \lambda = 100, \quad V = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}, \quad \mu = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad r_f = 1$$

2. Let y denote our current position in the risky assets. Suppose the transaction cost is a quadratic function of the risky asset bought or sold. In particular, the investor needs to pay an extra amount of $\gamma(x_i - y_i)^2$ in order to change her position in asset i from y_i to x_i. Please write down a new quadratic programming problem for the portfolio selection problem including the transaction cost. In particular, it should of the form

$$\begin{array}{rl} \min & f^T x + \frac{1}{2} x^T H x, \\ \text{s.t.} & Ax \le b \\ & A_{eq} x = b_{eq} \\ & l \le x \le u \end{array}$$

   Identify H, f, A_{eq}, b_{eq}, A and b.

Extra Paper 5

Problem 6

Suppose the following are $N = 10$ samples of losses on a portfolio.

$$\{0.0015 \quad 2.5892 \quad -0.1884 \quad 2.7155 \quad 0.0002 \quad -4.2795 \quad 0.1135 \quad 0.0926 \quad -3.6226 \quad 0.2430\}.$$

Set $\alpha = 0.65$.

1. Compute VaR_α and CVaR_α for the above loss sequence.
2. Suppose there are n assets in the market, and the rate of losses of these asset depend on m risk factors $\tilde{X} \in \mathbb{R}^m$, i.e. $L_i = \phi_i(\tilde{X})$, $i = 1, \ldots, n$, where ϕ_i are specified functions and are known.

   We want compute

   - a portfolio $\mathbf{w}$ of these assets such that $w_i \geq -\kappa$ for $i = 1, \ldots, n$,
   - that maximizes the expected return of the portfolio
   - subject to a constraint that the value-at-risk at level α of the rate of loss on the portfolio is at most β.

   Show how you would approximate this optimization problem by integer linear program by taking samples of $\tilde{X}$. You can assume that you have access to a random number generator that generates samples of $\tilde{X}$. Be very careful about how you define and compute the constant M that you use in your formulation.

Extra Paper 6

Problem 7

Let $p_1(t,x)$ and $p_2(t,x)$ be European put options on the same underlying S with the same expiry T, where p_1 has strike $K_1 > 0$, and p_2 has strike $K_2 > K_1$. Consider a portfolio long on p_2 and short on p_1, known as a **bear put spread** p_{spr}. Prove that the delta $\Delta_{spr}(t,x) \equiv \frac{\partial}{\partial x} p_{spr}(t,x)$ of the spread is negative for all $t < T, x > 0$.

Extra Paper 7

Problem 8

Let $W(t)$ be Brownian motion on $[0, T]$. Let $\Pi = \{t_0, ..., t_n\}$ be a partition of $[0, T]$, and define the **sampled quadratic variation** corresponding to Π:

$$Q_\Pi = \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2 .$$

Prove that $\lim_{\|\Pi\| \to 0} Q_\Pi = T$, where $\|\Pi\| = \min_i(t_{i+1} - t_i)$, and the convergence is in quadratic mean.

Extra Paper 8

Problem 9

Let $\widetilde{W}(t)$ be Brownian motion in a probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$, where $\widetilde{\mathbb{P}}$ is risk-neutral and the risk-free rate is a constant $r > 0$. Let τ_m be the stopping time

$$\tau_m = \min(t \geq 0 : \widetilde{W}(t) = m),$$

where $m > 0$ and $\tau_m = \infty$ if $\widetilde{W}$ never reaches m. Compute the risk-neutral expected value $\widetilde{\mathbb{E}}[e^{-r\tau_m}]$ of the stopping-time discount factor $e^{-r\tau_m}$.

Extra Paper 9

Problem 10

Suppose the loss of a portfolio is normally distributed with mean μ and variance σ^2. Fix $\alpha \in (0,1)$. Compute the Expected Shortfall at confidence level α (ES_α) of the portfolio.

Extra Paper 10

Problem 11

Consider two independent random variables X, Y with density functions respectively. Demonstrate that the density function of the random variable $Z = X + Y$ is the convolution of the respective density functions $f_Z = f_X * f_Y$. Verify this proposition for two standard normal independent random variables

Extra Paper 11

Problem 12

In the Merton's model it is assumed that the value process evolves as a Geometric Brownian Motion. Assume that μ_V, $\sigma_V > 0$ are real constants and W_T is a standard Brownian motion. Therefore the value process evolves as:

$$V_0 \exp\left((\mu_V - \frac{1}{2}\sigma_V^2)T + \sigma_V W_T\right).$$

Default occurs when the final value is below the final value of debt B. Compute the derivative of the probability of default with respect to the value of the debt B.

Extra Paper 12