Qualifying Exam (January 2015): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.

Do 2 out of problems 4,5,6.

Do 3 out of problems 7,8,9,10,11,12,13,14.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature

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1). (a). Suppose an LP $\max\{z = c^t x \mid Ax = b,\ x \geq 0\}$, with a bounded feasible region, has l optimal extreme points $v_1, \ldots, v_l$. Prove that a feasible point is optimal to the LP if and only if it can be expressed as a convex combination of $v_1, \ldots, v_l$.

(b). Is the same true for an LP with an unbounded feasible region? In other words, is is true that a feasible point is optimal if and only if it can be expressed as a convex combination of the optimal extreme points $v_1, \ldots, v_l$? Prove or give a counterexample.

(c). Let $\{d_1, d_2, \ldots d_l\}$ be directions of unboundedness for the constraints $Ax = b,\ x \geq 0$. Prove that $d = \sum_{i=1}^{l} \alpha_i d_i$ with $\alpha_i \geq 0$ is also a direction of unboundedness for these constraints. (This part is not related to parts (a) and (b), above, of the problem.)

2). Consider the following two LPs: (P1) $\max\{c^t x \mid Ax \leq b\}$, and (P2) $\min\{c^t x \mid Ax \geq b\}$.

(a). Write the duals of the problems (P1) and (P2).

(b). Suppose both (P1) and (P2) are feasible. Prove that if one of them has a finite optimal solution then so does the other.

(c). Suppose both (P1) and (P2) are feasible. Prove that if one of them has an unbounded objective function then so does the other.

(d). Suppose that both (P1) and (P2) have finite optimal solutions. Let x_1 be a feasible point to (P1) and x_2 be a feasible point to (P2). Prove that $c^t x_1 \leq c^t x_2$.

3). Consider the transshipment problem on the following network: Node 1 has supply 3, node 3 has supply 2, node 2 has demand 4, node 4 has demand 1. $u_{1,2} = 2$, $u_{1,3} = 2$, $u_{3,2} = 4$, $u_{2,3} = 1$, $u_{2,4} = 3$, $u_{3,4} = 7$, $c_{1,2} = 2$, $c_{1,3} = -1$, $c_{3,2} = 2$, $c_{2,3} = 5$, $c_{2,4} = 3$, $c_{3,4} = 2$,

(a). Set up the Linear Programming problem for this transshipment problem.

(b). Let $(1,3)$ $(3,2)$ $(3,4)$ be tree arcs, $(1,2)$ a non-tree arc at upper bound, and $(2,4)$ $(2,3)$ non-tree arcs at lower bound (0). What is the feasible tree solution? (Give the values of the x_{ij}'s.)

(c). Find the fair prices for this feasible tree solution.

(d). Which arc enters the tree next? Find the next feasible tree, and the x_{ij}'s.

4). Let $\{X_n, n \geq 0\}$ be a discrete-time Markov chain with state space $\{0, 1\}$ and transition probability matrix

$$\begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix},$$

where $\alpha, \beta \in (0,1)$. Define $N(n)$ as the number of visits to state 0 up to time n. Assuming $X_0 = 0$, find the asymptotic distribution of $N(n)$.

5). Let $\{N(t), t \geq 0\}$ be a renewal process with continuous interrenewal time distribution G. Let $\{A(t), t \geq 0\}$ be the age process associated with $\{N(t)\}$, i.e., $A(t) = t - S_{N(t)}$, where S_n is the time of the occurrence of the nth event. For a given constant $x > 0$, derive a renewal-type equation for calculating $H(t) := P(A(t) \leq x)$ and compute the limiting distribution of $A(t)$.

6). Consider an M/M/1/K queue with arrival rate λ, service rate μ, and finite capacity $K = 4$. Suppose at time 0, there is a single customer in the system. Let T be the arrival time of the first customer who finds the system empty. Find $E[T]$.

7). Let S be a set of n line segments in the plane. A line ℓ that intersects all segments of S is called a *stabber* for S.

(a). How efficiently (in big-Oh) can one decide if a stabber exists for S? Explain briefly how the method works.

(b). Now assume that all segments S are vertical. How efficiently (in big-Oh) can one decide if a stabber exists? Explain briefly.

(c). Now suppose we want to preprocess a set S of n line segments in the plane into a data structure so that when you are given a query line ℓ you can quickly report how many segments of S the line ℓ intersects. Give the preprocessing time, the storage space (for the data structure) and the query time (in big-Oh), and explain briefly.

(d). Suppose in part (c) that we are told that ℓ will definitely be of slope 1 or -1 or 2 or -2. What preprocessing time, space, and query time can now be achieved?

8). Let P be a simple n-gon in the plane.

(a). Suppose G is a *minimal guard set* within P: G is a set of points $g_i \in P$ so that every point of P is seen by at least one point of G (i.e., G is a valid guard cover of P), *and* the set G is *minimal*, meaning that deletion of any one point from G will cause G to stop being a valid guard cover of P). Give an example showing that G can have at least 5 times as many points as has a *minimum* guard cover G^* (a set of points that is a valid guard cover of P and has the fewest points of any guard cover of P).

(b). How efficiently can one compute a set G of at most $n/2$ points within P so that G is a valid guard cover of P?

(c). The following algorithm has been proposed to compute a set G of at most $n/2$ points within P (in fact, vertices of P) so that G is a valid guard cover of P: Walk through the vertices of P, classifying each as "convex" or "reflex"; let C be the set of convex vertices and let R be the set of reflex vertices; we know that at least one of the sets R or C has at most $n/2$ points – place guards at these points. (You may assume that the vertices of P are in general position – no three are collinear.)

(i). How efficient is this algorithm? (in big-Oh)

(ii). Does the algorithm work (to give a valid guard set of at most $n/2$ gusrds)? If yes, explain briefly why; if no, give a counterexample.

9). You are a real estate agent trying to sell a house. The owner of the house has authorized you to accept and reject offers. At the beginning of each month, if the house has not yet been sold, you must advertise the house. Advertising incurs administrative costs of A dollars per month. During the month, various offers will come in from interested buyers, and at the end of the month you can choose to either sell the house at the highest bid value or not sell the house that month, in which case you will advertise the house again in the following month. All bids expire at the end of the month, and you may assume you will receive a completely new set of bids each month. The probability that the highest bid is D dollars in any month is $p(D)$. Assume that the bids may take any of the K given values. If you accept a bid you will receive 10% of the bid value as commission. If you have not sold the house after N months, the house will be given to another realtor to sell, and you will receive nothing.

(a) Formulate this problem as a Markov decision process and write the optimality equations.

(b) Explicitly solve the problem as stated for $N = 3$, $A = 500$, $p(200,000) = 0.3$, $p(300,000) = 0.3$, and $p(0) = 0.4$, where $p(0)$ as the probability that no offer is presented. Find an optimal policy and your optimal expected profit.

10). For $\epsilon > 0$, does a stationary ϵ-optimal policy exist for a Markov Decision Process with countable state and action sets? Answer this question for discounted and average-cost problems. Justify your answers.

11). Consider a directed graph with positive arc lengths and a node s. Let T be a shortest path tree rooted at s. Define the *tolerences* of an arc (i, j) as the maximum increase α_{ij}, and the maximum decrease β_{ij}, that the arc length can tolerate without changing the shortest path tree rooted at s.

(a). Show that if the arc $(i, j) \notin T$ then $\alpha_{ij} = \infty$. Describe an efficient method to compute β_{ij}. What is the running time of your algorithm?

(b). Describe an efficient method to compute α_{ij} for $(i, j) \in T$. What is the running time of your algorithm?

12). Consider a directed graph $G = (V, A)$ $s, t \in V$, with arc capacities u_{ij} that are integral or infinite.

(a). Prove that v, the max flow from s to t, is finite if and only if there is no directed path from s to t containing only arcs of infinite capacity.

(b). Now suppose that there are no infinite capacity paths from s to t. Let A^0 be the set of arcs with finite capacity, and let $M = \sum_{(i,j)\in A^0} u_{ij}$. Show that replacing the capacity of each infinite capacity arc by M does not change the value of the max flow v.

13). Let X be a non-negative integer-valued random variable with hazard-rate function $h(k) = P(X = k | X \geq k)$. Let $U_1, U_2, \ldots$ be a sequence of i.i.d. random variables uniformly distributed over $[0, 1]$. Show that $Y = \min\{n : U_n \leq h(n)\}$ has the same distribution as X and hence give an algorithm for generating i.i.d. realizations of X.

14). Given an algorithm for generating random variates from the density function

$$f(x) = \frac{1}{\pi\sqrt{1 - x^2}} + 3x(1 - x),\ 0 \leq x \leq 1.$$