BENDING OF LIGHT AROUND A BLACK HOLE Neil P. Patel and Madappa Prakash, Department of Physics & Astronomy, State University of New York at Stony Brook

Observational tests of the predictions of Einstein's theory of "General Relativity" concerning the periastron shifts of the planets in our solar system have lent strong support to the notion that in the vicinity of a black hole, which has a very strong gravity field, photons will bend to a very high degree or even spiral into the black hole [1]. In this project, we calculate a light ray's orbit around a non-rotating and chargeless (Schwarzschild) black hole for several values of the impact parameter b and the mass M of the black hole. Such orbits are also characteristic of the nearly massless neutrinos in the vicinity of a black hole.

In the following, we adopt geometrized units by setting $G/c^2 = 0.742 \times 10^{-28}$ cm/g, where G is Newton's gravitational constant and c is the speed of light. In these units, the solar mass $\mathrm{M}_\odot = 1.989 \times 10^{33}$ g translates to $\mathrm{M}_\odot \cong 1.47$ km.

In the general relativistic Schwarzschild metric, the laws of conservation of energy and angular momentum predict that the orbit of a photon obeys the differential equation [1]

$$\left(\frac{du}{d\phi}\right)^2 + (1-2u)u^2 = \left(\frac{M}{b}\right)^2 = \frac{1}{\tilde{b}^2}\,, \tag{1}$$

where $u = M/r$. Differentiating both sides with respect to ϕ, one obtains

$$\frac{d^2u}{d\phi^2} + u = 3u^2 \quad \text{or} \quad u'' + u = 3u^2\,. \tag{2}$$

Before analyzing the general solutions, we illustrate how the periastron shifts are obtained. For $u << 1$ (far away from the black hole), the term $3u^2$ can be neglected in a zeroth order approximation to u which is obtained from $u_0'' + u_0 = 0$ for which $u_0 = (M/b)\cos\phi$ is a solution that describes a straight line path with no deflection. Corrections to this solution (which is inappropriate near a black hole) may be obtained by setting $u = u_0 + u_1$, where u_1 stands for how far off u is from u_0. Inserting this ansatz in Eq. (2) yields

$$u_1'' + u_1 = 3(u_0^2 + 2u_0u_1 + u_1^2)\,. \tag{3}$$

Since we are seeking small corrections, terms containing u_1 on the right hand side of the above equation may be neglected to get $u_1'' + u_1 = 3u_0^2$ for which

$$u = u_0 + u_1 = (M/b)\cos\phi + (M/b)^2[2 - \cos^2\phi] \tag{4}$$

is the solution. The above quadratic equation for $\cos^2\phi$ is straightforward to solve. Finding the two angles at which $u = 0$ and $r = M/u \to \infty$ yields a deflection of

$$\Delta\phi = 4M/b\,. \tag{5}$$

Observed periastron shifts of the planets in our solar system confirm the predictions of theory [1].

Photon orbits for impact parameters that lie close to the black hole require a numerical integration of Eq. (2), since with few exceptions analytical solutions cannot be found in terms of elementary functions. The second order non-linear differential equation in Eq. (2) is equivalent to the following two first order, but coupled, differential equations:

$$f = \frac{du}{d\phi} = y \qquad \text{and} \qquad g = \frac{d^2u}{d\phi^2} = \frac{dy}{d\phi} = 3u^2 - u \,. \tag{6}$$

The appropriate initial conditions are

$$u(\phi = 0) = \frac{M}{r_{TP}} \quad \text{and} \quad \frac{du}{d\phi}(\phi = 0) = 0 \,, \tag{7}$$

where r_{TP} denotes the turning point at which $\tilde{b}^{-2} = (1 - 2u)u^2$. In what follows, we have utilized a fourth-order Runge-Kutta scheme to find the solutions of Eq. (6). Explicitly, the updating of the solution is performed by the iterative scheme

$$\begin{aligned}
f1 &= hf(t, x, y)\,, \qquad g1 = hg(t, x, y) \\
f2 &= hf(t + h/2, x + f1/2, y + g1/2)\,, \qquad g2 = hg(t + h/2, x + f1/2, y + g1/2) \\
f3 &= hf(t + h/2, x + f2/2, y + g2/2)\,, \qquad g3 = hg(t + h/2, x + f2/2, y + g2/2) \\
f4 &= hf(t + h, x + f3, y + g3)\,, \qquad g4 = hg(t + h, x + f3, y + g3) \\
xn &= x + (f1 + 2(f2 + f3) + f4)/6 \\
yn &= y + (g1 + 2(g2 + g3) + g4)/6
\end{aligned} \tag{8}$$

where, for the problem on hand, we let $\phi \to t$, $u \to x$, and $u' \to y$. We have checked that a sixth-order Runge-Kutta scheme gives nearly the same results. Figure 1 shows the orbits for the indicated values of r_{TP}/M. Among the main points to note from these results are that

• For $r_{TP}/M = 3$, the photon orbits the black hole in an unstable circular orbit that is called the photon sphere whose radius is 1.5 times the Schwarzschild radius $2GM/c^2$;
• For $r_{TP}/M < 3$, the photon spirals into the black hole; and
• For $r_{TP}/M \gg 9$, the orbit approaches a straight line, but with a total deflection of $4M/b$.

The behavior of light and the nearly massless neutrinos in the vicinity of compact objects such as black holes and very compact neutron stars is becoming increasingly important in the interpretation of the evergrowing observations of such objects using satellites such as the Hubble Space Telescope, the Chandra and the XMM-Newton. We intend to continue these studies in an effort to increase our understanding of these high gravity environments.

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References

[1] C.W. Misner, K. S. Thorne, & J.A. Wheeler, *Gravitation* (W.H Freeman & Company, NY, 1973).

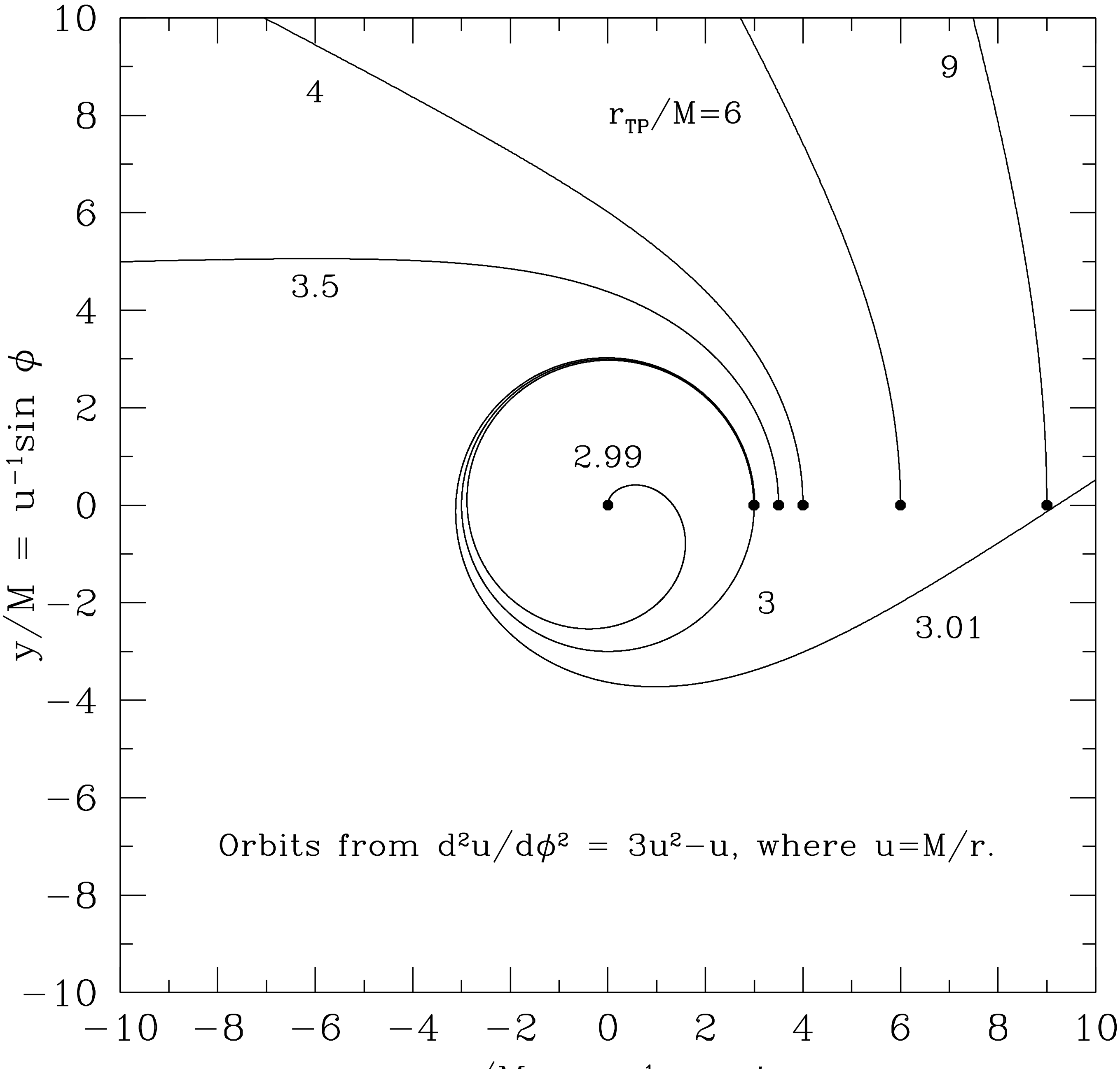

Figure 1: Photon orbits around a black hole obtained from a sixth-order Runge-Kutta scheme to Eq. (6).