Proper Welfare Weights for Social Optimization Problems

Alexis Anagnostopoulos, Eva Carceles-Poveda, Yair Tauman

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Abstract. Social optimization problems are often used in economics to study important issues. In a social optimization problem, the sum of individual weighted utilities is maximized over all feasible allocations that satisfy certain constraints. In this paper, we provide a mechanism that determines the set of proper individual weights to be applied to social optimization problems. To do this, we first define for every set of individual weights and for every social welfare function the contribution of every individual to the total welfare through the individual's initial endowments. We then provide an axiomatic approach to the notion of the per unit contribution of every good and every individual. We then define a set of individual weights to be proper iff the weighted utilities of every individual from this allocation are proportional to the contribution of the individual to the total welfare as defined by this set of weights. It is shown that every contribution mechanism that satisfies these four axioms is uniquely determined by a non negative measure on the unit interval. The selection of a specific contribution mechanism (or equivalently the selection of a specific nonnegative measure on the unit interval) determines for a given economy and a given set of weights a proper constrained efficient allocation and a proper set of weights. Finally, we provide several numerical examples that illustrate our methodology. When households are not ex ante identical, the examples suggest that using the proper weights can substantially affect the allocations.

Keywords: Welfare weights, Heterogeneous Households

JEL Classification: D52, D63

1. Introduction

Social optimization problems are often used in economics to study important issues. In a social optimization problem, the sum of individual weighted utilities is maximized over all feasible allocations that satisfy certain constraints. The allocations resulting from this problem are thus constrained efficient. Moreover, every set of individual welfare weights defines a social welfare function and hence determines a constrained efficient allocation. Under standard assumptions, the converse is also true, namely, every constrained efficient allocation maximizes the sum of weighted utilities for some set of individual weights.

Under complete markets, the welfare theorems imply that equilibria are pareto optimal, suggesting that one can select the particular set of individual weights for a social optimization problem as the ones that implement competitive equilibria. Under incomplete markets, however, it is not obvious how to determine these weights, particularly when individuals are not ex ante identical. The purpose of this paper is to provide a mechanism that determines the set of proper individual weights to be applied to social optimization problems. To do this, we first define for every set of individual weights and for every social welfare function the contribution of every individual to the total welfare through the individual's initial endowments. We then provide an axiomatic approach to the notion of the per unit contribution of every good and every individual, where the contribution of an individual to the total welfare is the total contribution of his initial endowments. We then define a set of individual weights to be proper iff the weighted utilities of every individual from this allocation are proportional to the contribution of the individual to the total welfare as defined by this set of weights.

The axiomatic approach consists of four axioms that characterize an elegant family of contribution mechanisms. The first axiom asserts that the per unit contribution should be independent of the units of measurement of the goods. The second asserts that if two (or more) goods play the same role in the welfare function, they should have the same per unit contribution. The third axiom asserts that if the welfare function can be broken into different components, then the per unit contribution of a given good is the sum of the per unit contributions arising from the different components. The last axiom guarantees that the per unit contribution is a continuous mapping with respect to an appropriate norm.

It is shown that every contribution mechanism that satisfies these four axioms is uniquely determined by a non negative measure on the unit interval. The selection of a specific contribution mechanism (or equivalently the selection of a specific nonnegative measure on the unit interval) determines for a given economy and a given set of weights a proper constrained efficient allocation and a proper set of weights. Last, we suggest two variants of an additional fifth axiom. The first variant uniquely determines the Lebesque measure on the unit interval and the second one uniquely determines the measure which concentrates the total mass on the last point of the unit interval (one).

Finally, we provide several numerical examples that illustrate our methodology. In the first example, we compute the constrained efficient allocations in a two period model with incomplete markets and ex-ante different households and we use our method mechanism to compute proper allocations and weights. In the second, we compute the optimal taxes of a Ramsey economy when households are ex ante different and we find the proper individual weights.

2. The Economy

Consider an economy with K goods and n agents that are indexed by $i \in I \equiv \{1, 2, ..., n\}$. In what follows, we let $w^i = \left(w_1^i, ...w_K^i\right) \in \mathbb{R}_+^K$ be the vector of initial endowments of agent $i \in I$, while $\left(w^1, ..., w^n\right) \in \mathbb{R}_+^{Kn}$ is vector of individual endowments. Similarly, $x^i = \left(x_1^i, ..., x_K^i\right) \in \mathbb{R}_+^K$ represents the vector of consumptions of agent $i \in I$ and $\left(x^1, ..., x^n\right)$ is the vector of individual consumptions. Let $u_i\left(x^i\right)$ represent the utility function of agent $i \in I$.

Assumption 1. u_i is continuously differentiable and concave on $\mathbb{R}_+^K$ with $u_i(0) = 0$ for all $i \in I$.

Definition 1. Social Optimization Problem. A social optimization problem (SOP) is one that maximizes the weighted sum of utilities u_i under certain constraints, where the individual weights $\lambda = (\lambda_i)_{i \in I}$ satisfy $\lambda_i \geq 0$ for all $i \in I$ and $\sum_{i=1}^n \lambda_i = 1$. The resulting value function $F_\lambda\left(w^1, ..., w^n\right)$ is a social welfare function. Formally,

$$
\begin{aligned}
F_\lambda\left(w^1, ..., w^n\right) &\equiv \max \sum_{i=1}^n \lambda_i u_i\left(x^i\right) \text{ s.t.} \\
x^i &\in \mathbb{R}_+^K, \; i = 1, ..., n \\
h_1(x, w) &\geq 0 \\
&\vdots \\
h_s(x, w) &\geq 0
\end{aligned}
\tag{1}
$$

For example, in a pure exchange economy with complete markets, a typical social welfare function is of the form:

$$
F_\lambda(w_a) = \max_{(x^i)_{i \in I}} \sum_{i=1}^n \lambda_i u_i\left(x^i\right) \text{ s.t. } x_a = w_a \tag{2}
$$

where $w_a = \sum_{i=1}^n w^i \in \mathbb{R}_+^K$ and $x_a = \sum_{i=1}^n x^i \in \mathbb{R}_+^K$ are the vectors of aggregate endowments and allocations respectively. In this example, the constraint is linear and F_λ is a function of aggregate endowments only (the distribution of endowments is unimportant). Hence, we can write $F_\lambda(w_a)$ instead of $F_\lambda\left(w^1, ..., w^n\right)$. It is easy to verify that $F_\lambda(w_a)$ is concave and continuously differentiable on $\mathbb{R}_+^K$, with $F_\lambda(0) = 0$. In other SOPs[1], however, the constraints might not be linear or convex and the social welfare function may not be concave or continuously differentiable.

This paper provides a method to compute the contribution of a good and/or an individual to aggregate welfare. The method is justified by an axiomatic approach described in detail in the following section. Here, we take the contributions as given and provide an application that illustrates the usefullness of those contributions. In particular, we consider the question of determining the welfare weights λ_i to be used in the SOP. We postulate the following

[1] For example, when markets are incomplete or in optimal Ramsey taxation problems.

principle: welfare should be allocated to individuals according to their contribution and the welfare weights are chosen to satisfy this principle. Specifically, we will define *proper* welfare weights (and corresponding *proper* allocations) to be such that fraction of aggregate welfare received by an individual equals the fraction the individual contributes.

To make these statements more precise, we introduce some notation. Let $c_k^i(F_\lambda, w)$ be the per unit contribution of good $k = 1, ..., K$ held by agent $i \in I$ to social welfare F_λ given initial endowments w.[2] Define the total contribution of agent $i \in I$ to be $w^i \bullet c^i(F_\lambda, w) \in \mathbb{R}$.

Definition 2. Proper Allocation and Weights. Consider a SOP with weights $\lambda = (\lambda_i)_{i \in I}$ and let $x^*(\lambda) = \left(x^{*1}(\lambda), ..., x^{*n}(\lambda)\right) \in \mathbb{R}_+^{Kn}$ be a maximizer. The weights λ and the allocation $x^*(\lambda)$ are proper iff

$$\frac{u_i\left(x^{*i}(\lambda)\right)\lambda_i}{u_j\left(x^{*j}(\lambda)\right)\lambda_j} = \frac{w^i \bullet c^i(F_\lambda, w)}{w^j \bullet c^j(F_\lambda, w)} \tag{3}$$

That is, the proper weights and allocations are such that the weighted utility of i relative to j must equal the relative contributions of i and j.

Clearly, $x^*(\lambda)$ is proper iff

$$\frac{u_i\left(x^{*i}(\lambda)\right)\lambda_i}{\sum_{j \in I} u_j\left(x^{*j}(\lambda)\right)\lambda_j} = \frac{w^i \bullet c^i(F_\lambda, w)}{\sum_{j \in I} w^j \bullet c^j(F_\lambda, w)} \tag{4}$$

Our next goal to define properly the contribution of every agent to the total welfare.

3. Axiomatic Approach for the Contribution Mechanism

In this section, we develop an axiomatic approach for the contribution mechanism. To simplify notation, we let $m = Kn$ be the total number of goods. Goods are differentiated along two dimensions depending on the type of good ($k = 1, ..., K$) and the owner of the good ($i \in I$). Let $w = (w_1, ..., w_m)$ be the vector of initial endowments, where $j = (i-1)K + k$ represents the initial endowment of agent $i \in I$ and good $k = 1, ..., K$. In what follows, we suppress the subindex λ and write $F(w)$ for a typical social welfare function.

We denote by $\mathbb{R}^m$ the m dimensional Euclidian space, by $\mathbb{R}_+^m$ the nonnegative orthant and by $\mathbb{R}_{++}^m$ the positive orthant of $\mathbb{R}^m$. We let $\mathcal{F}^m$ be the set of all social welfare functions F which are defined on $\mathbb{R}_+^m$ such that (1) F is continuously differentiable on $\mathbb{R}_+^m$ and (2) $F(0) = 0$. Namely, F can be any continuously differentiable function which arises from a SOP of some economy.

In general, we do not know how to characterize the set $\mathcal{F}^m$ of social welfare functions. This set depends on the economies under consideration and, in particular, on the set of constraints of each economy. However, if the set of economies includes the pure exchange economies with complete markets, we can easily prove (see Lemma 1 below) that $\mathcal{F}^m$ contains $\mathcal{F}_{con}^m$, where $\mathcal{F}_{con}^m$ is the set of all functions on $\mathbb{R}_+^m$ such that (1) F is continuously differentiable and concave on $\mathbb{R}_+^m$ and (2) $F(0) = 0$. In other words, the lemma states that every continuously differentiable function which is concave and satisfies $F(0) = 0$ is a social welfare function.[3]

[2] Note that, in principle, the per unit contribution of the same good can differ depending on which individual holds it.

[3] The proof of Lemma 1 as well as all other results are provided in the Appendix, unless otherwise stated).

Lemma 1. $\mathcal{F}^m_{con} \subseteq \mathcal{F}^m$.

Our next goal is to define for each $F \in \mathcal{F}^m$ and for each $j = 1, ..., m$ the per unit contribution of every good and of every agent $i \in I$ to the total welfare $F(w)$.

Definition 3. Contribution Mechanism. A contribution mechanism is a rule or a function $c(\cdot,\cdot)$ which associates with every m, with every $F \in \mathcal{F}^m$ and every $w \in \mathbb{R}^m_+$ an element $c(F,w)$ in $\mathbb{R}^m$, where

$$c(F,w) = (c_1(F,w), ..., c_m(F,w))$$

and $c_j(F,w)$ for $j = 1, ..., m$ is the per unit contribution of the j^{th} good to the total welfare of society.

Below, we present four requirements (axioms) that restrict the set of contribution mechanisms to an elegant family.

Axiom 1: Rescaling. Let $G \in \mathcal{F}^m$. Let $r \in \mathbb{R}^m_{++}$ and let $F \in \mathcal{F}^m$ be defined by

$$F(x_1, ..., x_m) \equiv G(r_1 x_1, ..., r_m x_m)$$

Then, for each $w \in \mathbb{R}^m_+$ and each $j = 1, ..., m$

$$c_j(F,w) = r_j c_j(G, r_1 x_1, ..., r_m x_m)$$

This axiom requires that the contributions should be independent of the units of measurement of the goods. The idea can be illustrated in a simple one good economy. Consider an economy where only apples are consumed, let x denote apple allocations in kilos and let the social welfare function be $F(x)$. Let $G(x)$ denote the social welfare function of an identical economy where x is measured in grams. Clearly, $F(x) = G(1000x)$. Then, $c(F,\alpha)$ is the per kilo contribution in an economy initially endowed with α kilos of apples. On the other hand, $c(G, 1000\alpha)$ is the per gram contribution in an economy initially endowed with 1000α grams (=α kilos) of apples. Since the two economies have the same initial endowment and are identical otherwise, the axiom requires that the per kilo contribution equals 1000 times the per cent contribution

$$c(F,\alpha) = 1000c(G, 1000\alpha)$$

Axiom 2: Consistency. Suppose that $F \in \mathcal{F}^m$ and $G \in \mathcal{F}^1$. If for every $x \in \mathbb{R}^m_+$

$$F(x) = G\left(\sum_{k=1}^{m} x_k\right)$$

then for each $j = 1, ..., m$ and for each $w \in \mathbb{R}^m_+$

$$c_j(F,w) = c\left(G, \sum_{k=1}^{m} w_k\right)$$

This axiom asserts that if two (or more) goods are the "same", in the sense that they play the same role in the welfare function, they should have the same per unit contribution. For example, suppose that there are two goods, say blue and red cars. Assume that the welfare associated with a blue car is the same as the one associated with a red car and let G be a one variable function indicating the welfare of a total number of cars (regardless of whether they are blue or red). Then the previous statement can be formulated as follows: $F(x_1, x_2) = G(x_1 + x_2)$. In this case, the axiom requires that the contribution of a blue car is the same as the one of a red car, namely $c_1(F, (x_1, x_2)) = c_2(F, (x_1, x_2)) = c(G, x_1 + x_2)$.

Axiom 3: Additivity. Suppose that F, G and H belong to $\mathcal{F}^m$. If for every $x \in \mathbb{R}_+^m$

$$F(x) = G(x) + H(x)$$

then for every $w \in \mathbb{R}_+^m$

$$c(F, w) = c(G, w) + c(H, w)$$

This axiom asserts that if the welfare function F can be broken into the sum of two components G and H, then the per unit contribution of the j^{th} good is the sum of the per unit contributions arising from G and H.

Axiom 4: Positivity. Let F and G be in $\mathcal{F}^m$ and let $w \in \mathbb{R}_{++}^m$. Suppose that $F - G$ is non decreasing for all $0 \leq x \leq w$, where $x \leq w$ means that $x_j \leq w_j$ for all $j = 1, ..., m$. Then

$$c(F, w) \geq c(G, w)$$

The idea in Axiom 4 is that, if increasing initial endowments results in a larger increase in welfare in economy F compared to economy G, then it must be that the per unit contribution in the former is larger than in the latter. Given the additivity axiom, the positivity axiom guarantees that for every $w \in \mathbb{R}_+^m$ the per unit contribution $c(\cdot, w)$ is continuous in the C^1-norm.

Theorem 1 characterizes the set of all contribution mechanisms that satisfy the previous four axioms.

Theorem 1. $c(\cdot, \cdot)$ is a contribution mechanism on $\mathcal{F} = \cup_{m=1}^{\infty} \mathcal{F}^m$ which satisfies Axioms 1-4 iff there exists a unique nonnegative measure μ on $([0,1], ß)$ ($ß$ is the set of all Borel subsets of $[0,1]$) such that for each m, for each $F \in \mathcal{F}^m$ and for each $w \in \mathbb{R}_+^m$ with $w \neq 0$,

$$c_j(F, w) = \int_0^1 \frac{\partial F}{\partial x_j}(tw)\, d\mu(t), \; j = 1, ..., m \tag{5}$$

That is, (5) defines a one to one mapping from all the nonnegative measures on $([0,1], ß)$ onto the set of all contribution mechanisms satisfying the four axioms.

For an intuitive interpretation of the formula (5) suppose that there is only one good and the initial endowment is w. One can compute the increase in welfare resulting from a marginal increase in the initial endowment at zero. This computation can be carried out for successively larger initial endowments until w is reached. In this way we will have the marginal addition in welfare arising from each additional unit of endowment at every step of the way. The contribution of the good to the welfare function will be the average of these marginal effects weighted by the measure μ. In mutli-good environments, these computations will be made on the diagonal (i.e. only at points where the endowments of different goods are at a fixed proportion).

The axiomatic approach, as it stands, does not specify what measure μ should be used. However, slight modifications of the axioms can uniquely determine this measure. In what follows, we provide two such modifications. The first one adds the requirement that the contributions add up to the total welfare, Axiom 5. In that case, the unique measure μ satisying the axioms is the Lebesgue measure on $[0, 1]$, according to which the formula in (5) becomes

$$c_j (F, w) = \int_0^1 \frac{\partial F}{\partial x_j} (tw)\, dt,\ j = 1, ..., m$$

This measure would average the marginal effects on welfare along the diagonal $[0, w]$. The second one replaces positivity (Axiom 4) with a stronger requirement (Axiom 4'). In that case, the unique measure is the atomic probability measure whose whole mass is concentrated at the point $t = 1$. The contribution mechanism specifies that, for any $F \in \mathcal{F}^m$ and any $w \in \mathbb{R}^m_+$,

$$c_j (F, w) = \frac{\partial F}{\partial x_j} (w)\, ,\ j = 1, ..., m$$

According to this measure, the contribution of the jth good is simply equal to its marginal effect on welfare at w. The additional axioms together with the resulting Propositions are presented below.

Axiom 5. Welfare Allocation. For every $F \in \mathcal{F}^m$ and every $w \in \mathbb{R}^m_+$

$$\sum_{j=1}^{m} w_j c_j (F, w) = F (w)$$

This axiom asserts that the sum of contributions should be equal to the total welfare.

Proposition 1. There exists a unique contribution mechanism which satisfies axioms A1, A2, A4 and A5. The measure μ is the Lebesgue measure on $[0, 1]$ and the contribution mechanism is:

$$c_j (F, w) = \int_0^1 \frac{\partial F}{\partial x_j} (tw)\, dt$$

For a proof of Proposition 1, see Young (1985).

Axiom 4'. Let $F \in \mathcal{F}^m$ and let $w \in \mathbb{R}^m_{++}$. If F is non decreasing at each $x \leq w$ in a neighborhood of w then $c_j (F, w) \geq 0$, for $1 \leq j \leq m$.

This axiom requires that contributions are nonnegative at w even if F is nondecreasing only in a neighborhood of w. Note that axiom 4' implies axiom 4 so that by theorem 1 a contribution mechanism that satisfies axioms 1,2,3 and 4' is of the form (5). But the available set of mechanisms is now smaller.

Proposition 2. A contribution mechanism $c(\cdot,\cdot)$ satisfies Axioms A1-A3 and A4' iff there exists a constant $a \geq 0$ s.t. for each m, each $F \in \mathcal{F}^m_{con}$ and each $w \in \mathbb{R}^m_+$ with $w \neq 0$

$$c(F,w) = a\nabla F(w)$$

where $\nabla F(w)$ is the gradient of F evaluated at w. Namely, for each j with $1 \leq j \leq m$

$$c_j(F,w) = \int_0^1 \frac{\partial F}{\partial x_j}(tw)\,d\mu(t)$$

where $\mu([0,1]) = 0$ and $\mu(1) > 0$. The measure μ is an atomic measure which is fully concentrated at $t = 1$.

For a proof of Proposition 2, see Samet and Tauman (1982).

Axiom 5 is natural when the question is the allocation of welfare. Axiom 4' on the other hand is quite strong. The interest in this one arises from the fact that it yields a contribution mechanism and proper allocations that are familiar. Specifically, in a context where the fundamental welfare theorems apply, the proper allocation will be Walrasian with respect to the initial endowments. Put differently, our method of determining the welfare weights will yield the same answer as the well-known Negishi method. This is reassuring and provides further support to the use of proper weights when the welfare theorems do not apply and the Negishi method cannot be applied.

A similar result can be obtained for the case of the Lebesgue measure (under Axiom 5), albeit under the additional assumption of homogeneity. This is the objective of the following section.

4. Characterization of Proper Allocations in Homogeneous Economies

This section characterizes proper allocations in economies where the individual utilities and the constraints are homogeneous. We consider pure exchange economies with complete markets and show that in these markets a constrained efficient allocation is proper iff it is a Walrasian allocation (a competitive equilibrium allocation with respect to the initial endowments). In a later section, we will provide examples of proper allocations for non-homogeneous economies (whether these have complete or incomplete markets).

Proposition 3. Consider a pure exchange economy with complete markets. Suppose that the utilities $(u_i)_{i\in I}$ are all homogeneous of degree r for some $r > 0$. Then an allocation is proper iff it is a Walrasian (competitive) equilibrium with respect to the initial endowments w.

5. Applications

We are interested in economies with incomplete markets where, to our knowledge, there is no currently available method for determining welfare weights. The objective of these examples is illustrative, with two goals in mind. To show how our methodology can be applied and to hint on the potential importance of the choice of weights for economic applications. We consider an optimal taxation problem under incomplete markets where households are not ex ante identical. We focus on the Lebesgue measure.

5.1. The Economy. The model has two periods, $t = 1, 2$ and two types of agents that are indexed by $i \in \{1, 2\}$. In the first period, households receive an (certain) exogenous endowment y_i and they decide how much to consume c_{i1} and how much to save k_i. There is a continuum of households of type 1 (receiving y_1) of measure π_1 and a continuum of households of type 2 (receiving y_2) of measure $1 - \pi_1$. Thus households are not ex ante identical, unless $y_1 = y_2$. In the second period, households work and consume. Consumption c_{i2} equals income which arisesfromtwo sources, capital and labor. Capital income is Rk_i where R is the gross rateof return. Labor income is uncertain from the point of view of period 1, since the endowment of labor in period 2 e_i is stochastic. In particular, this endowment is high with probability π_2 and low otherwise:

$$e_i = \left\{ \begin{array}{c} e^H \text{ with prob. } \pi_2 \\ e^L \text{ with prob. } 1 - \pi_2 \end{array} \right\}$$

Households derive utility from consumption and from government expenditures g_2 which they treat as exogenous. Each period, households choose consumption demand as well as savings (capital), to maximize their expected utility subject to their budget constraints:

$$\max_{\{c_{i1}, c_{i2}, k_i\}} v(c_{i1}) + \beta \left[\pi_2 u(c_{i2}(e^H), g_2) + (1 - \pi_2) u(c_{i2}(e^L), g_2) \right]$$

subject to

$$\begin{aligned} c_{i1} + k_i &= y_i \\ c_{i2}(e^H) &= (1 - \tau_k) Rk_i + we^H \\ c_{i2}(e^L) &= (1 - \tau_k) Rk_i + we^L \end{aligned}$$

We assume the government levies proportional taxes on capital that are known to the household ex ante.Letting K denote aggregate capital and assuming a balanced government budget, $g_2 = \tau_k RK$. The aggregate capital supply in this economy is derived from the saving decisions of households as follows:

$$K \equiv \pi_1 k_1 + (1 - \pi_1) k_2$$

Moreover, aggregate labor supply is exogenous and given by

$$L \equiv \pi_2 e^H + (1 - \pi_2) e^L$$

Both aggregate capital and labor are rented by a single firm at prices R and w respectively. They are used as inputs to a constant returns to scale production function $F(K, L)$. The firm maximizes the period profits, leading to the following aggregate demand functions:

$$\begin{aligned} w &= F_L(K^D, L^D) \\ R &= F_K(K^D, L^D) \end{aligned}$$

5.2. Equilibrium. Given government policy (g_2, τ_k), the competitive equilibrium is characterized by the following conditions:

$$v'(c_{i1}) = \beta (1-\tau_k) R \left(\pi_2 u_c(c_{i2}\left(e^H\right), g_2) + (1-\pi_2) u_c(c_{i2}\left(e^L\right), g_2)\right)$$

$$\begin{aligned} c_{i1} + k_i &= y_i \\ c_{i2}(e^H) &= (1-\tau_k) Rk_i + we^H \\ c_{i2}(e^L) &= (1-\tau_k) Rk_i + we^L \\ \tau_k RK &= g_2 \end{aligned}$$

$$\begin{aligned} w &= F_L(\pi_1 k_1 + (1-\pi_1) k_2, \pi_2 e^H + (1-\pi_2) e^L) \\ R &= F_K(\pi_1 k_1 + (1-\pi_1) k_2, \pi_2 e^H + (1-\pi_2) e^L) \end{aligned}$$

together with goods market clearing which is satisfied by Walras'Law.

5.3. Ramsey Problem. We consider an optimal (Ramsey) taxation problem where the government chooses the optimal capital income tax (and government spending) in order to pick the competitive equilibrium that maximizes aggregate welfare. Aggregate welfare is a weighted average of individual welfare where each type's welfare is weighted by λ_i. The problem can thus be formulated as follows:

$$\max_{\{k_i, \tau_k\}} \sum_{i=1}^{2} \lambda_i \tilde{\pi}_i \left[v(y_i - k_i) + \beta \left[\pi_2 u((1-\tau_k) Rk_i + we^H, \tau_k RK) + (1-\pi_2) u((1-\tau_k) Rk_i + we^L, \tau_k RK)\right]\right]$$

where

$$\begin{aligned} \tilde{\pi}_1 \lambda_1 + \tilde{\pi}_2 \lambda_2 &= 1 \\ \tilde{\pi}_2 &= 1 - \tilde{\pi}_1 = 1 - \pi_1 \end{aligned}$$

s.t.

$$v'(y_i - k_i) = \beta (1-\tau_k) R \left(\pi_2 u_c((1-\tau_k) Rk_i + we^H, \tau_k RK) + (1-\pi_2) u_c((1-\tau_k) Rk_i + we^L, \tau_k RK)\right)$$

$$\begin{aligned} w &= F_L(\pi_1 k_1 + (1-\pi_1) k_2, \pi_2 e^H + (1-\pi_2) e^L) \\ R &= F_K(\pi_1 k_1 + (1-\pi_1) k_2, \pi_2 e^H + (1-\pi_2) e^L) \end{aligned}$$

5.4. Numerical Results. We make some assumptions on preferences and technology in order to obtain numerical results. We assume a linear production function

$$F(K, L) = RK + wL$$

implying that R and w are exogenous. To ensure that individual utility at zero equals zero, which is assumed in the previous sections, we choose an exponential utility[4]

$$\begin{aligned} v(c) &= 1 - e^{-c} \\ u(c, g) &= 1 - e^{-c} + \delta \left(1 - e^{-g}\right) \end{aligned}$$

[4] Our method can be easily applied to utilities that do not vanish at zero by applying a suitable transformation.

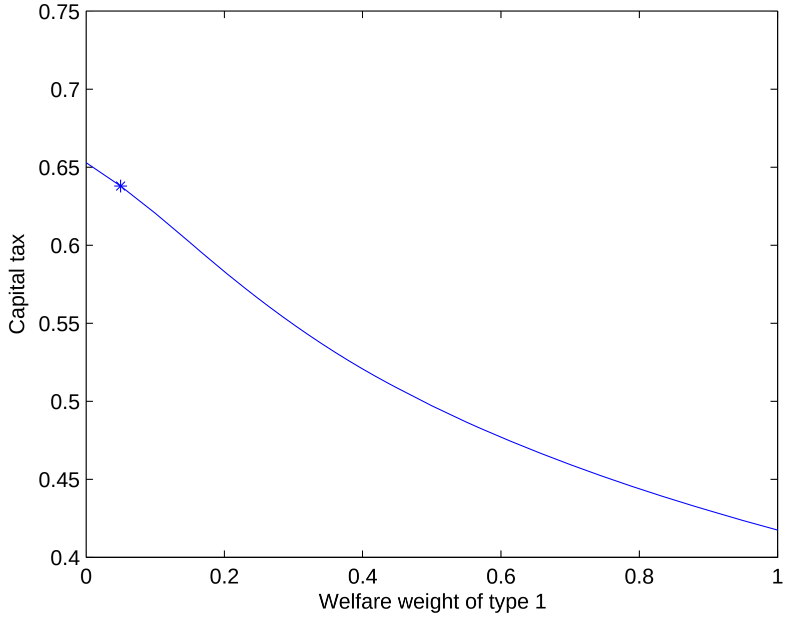

Figure 1:

where the parameter $\delta > 0$ governs the preference for the government good relative to the consumption good. Note that households of different types are assumed to have the same utility. Although it would be interesting to consider heterogeneity in preferences, for the purposes of this example we assume that households only differ in their initial endowments y_i.Table 1 below provides the parameter values used in the numerical computation.

Table 1: Parameter Values

β	δ	r	w	π_2	π_1	e^L	e^H	y_1	y_2
0.99	0.5	1.5	1	0.5	0.5	0.4	1.6	2	18

Note that welfare weights λ_i do not add up to one, so we normalize them and report their normalized values $\frac{\lambda_1}{\lambda_1+\lambda_2}$. Figure 1 below displays how the optimal tax varies as a function of the (normalized) welfare weight of type 1.

As reflected by the figure, the optimal tax varies significantly with the welfare weight of type 1 agents. In particular, the lower is the welfare weight of type 1, the higher is the capital income tax. In this particular example, the normalized weight on the poor turns out to be considerable lower than 0.5 and the optimal tax rate is around 0.64. The graph also reflects that if the government gives the same weight to both agent types then the optimal tax rate is equal to around 0.5. This example, while just being illustrative, suggests that choosing the same weights for all agents might strongly bias the results towards a lower optimal tax rate.

6. Conclusion

To be written.

Appendix A: Proofs

Proof of Lemma 1. Let $F \in \mathcal{F}^m_{con}$ and consider a pure exchange economy with m physical goods where for all $i \in I$ and for all $x \in \mathbb{R}^m_+$ $u_i(x) \equiv F(nx)$. Let w_a be the vector of aggregate initial endowments and let $\lambda_i = \frac{1}{n}$ for $i \in I$. Since agents have the same utility, we can write u instead of u_i. By the concavity of F

$$\sum_{i=1}^{n} \lambda_i u\left(x^i\right) = \sum_{i=1}^{n} \frac{1}{n} F\left(nx^i\right) \leq F\left(\sum_{i=1}^{n} \frac{1}{n} nx^i\right) = F\left(\sum_{i=1}^{n} x^i\right) = F\left(w_a\right)$$

Hence,

$$\begin{array}{c} \max_{(x^i)_{i\in I}} \sum_{i=1}^n \lambda_i u\left(x^i\right) \text{ s.t.} \\ \sum_{i=1}^n x^i_j = w_j,\ j = 1,...,K \end{array} \leq F\left(w_a\right)$$

On the other hand,

$$\begin{aligned} & \max_{(x^i)_{i\in I}} \sum_{i=1}^{n} \lambda_i u\left(x^i\right) \text{ s.t. } \sum_{i=1}^{n} x^i_j = w_j,\ j = 1,...,m \\ \geq\ & \sum_{i=1}^{n} \lambda_i u\left(\frac{1}{n} w_a\right) \text{ s.t. } \sum_{i=1}^{n} x^i_j = w_{aj},\ j = 1,..,m \\ =\ & \sum_{i=1}^{n} \lambda_i F\left(w_a\right) \text{ s.t. } \sum_{i=1}^{n} x^i_j = w_{aj},\ j = 1,...,m \\ =\ & F\left(w_a\right) \end{aligned}$$

Thus, $F \in \mathcal{F}^m$ implying that $\mathcal{F}^m_{con} \subseteq \mathcal{F}^m$.■

Proof of Theorem 1. Suppose that $c(\cdot,\cdot)$ is a contribution mechanism on $\mathcal{F} = \cup_{m=1}^{\infty} \mathcal{F}^m$ which satisfies Axioms 1-4. The idea of the proof is to extend the definition of $c(\cdot,\cdot)$ from $\mathcal{F}$ to $\mathcal{F}_{cd}$, where $\mathcal{F}_{cd} = \cup_{m=1}^{\infty} \mathcal{F}^m_{cd}$ and $\mathcal{F}^m_{cd}$ is the set of all continuously differentiable functions F on $\mathbb{R}^m_+$ such that $F(0) = 0$.[5] Then we prove (5) for every $F \in \mathcal{F}_{cd}$ and hence for every $F \in \mathcal{F}$.

By Lemma 1, we know that $\mathcal{F}^m_{con} \subseteq \mathcal{F}^m$ and hence $c(\cdot,\cdot)$ is defined on $\mathcal{F}^m_{con}$ for all $m = 1, 2, ...$We start by extending $c(\cdot,\cdot)$ from $\mathcal{F}^m_{con}$ to $\mathcal{F}^m_{con} - \mathcal{F}^m_{con}$, which is the set of all functions F which are the difference between any two functions in $\mathcal{F}^m_{con}$. We then show that every polynomial on $\mathbb{R}^m$ is in $\mathcal{F}^m_{con} - \mathcal{F}^m_{con}$. This will allow us to extend $c(\cdot,\cdot)$ to $\mathcal{F}_{cd}$ using the fact that every $F \in \mathcal{F}^m_{cd}$ can be approximated (in the C_1-norm) by polynomials on $\mathbb{R}^m$.

Suppose that $F = F_1 - F_2$, where $F_i \in \mathcal{F}^m_{con}$ for $i = 1, 2$. For all $w \in \mathbb{R}^m_+$ define

$$c(F, w) = c(F_1, w) - c(F_2, w) \tag{6}$$

We now show that (6) is well defined. Namely, it does not depend on the representatives F_1 and F_2. Suppose that F_1, F_2, G_1 and G_2 are all in $\mathcal{F}^m_{con}$ such that $F_1 - F_2 = G_1 - G_2$. We need to show that

$$c(F_1, w) - c(F_2, w) = c(G_1, w) - c(G_2, w) \tag{7}$$

[5] Note that not every $F \in \mathcal{F}^{cd}$ is a social welfare function. Furthermore, $\mathcal{F}^{cd}$ contains functions which may take negative values.

Note that $F_1 + G_2 \in \mathcal{F}^m_{con}$ and $G_1 + F_2 \in \mathcal{F}^m_{con}$. Since $F_1 + G_2 = G_1 + F_2$ we have that

$$c(F_1 + G_2, w) = c(F_2 + G_1, w)$$

By the additivity axiom on $\mathcal{F}^m_{con}$,

$$c(F_1, w) + c(G_2, w) = c(F_2, w) + c(G_1, w)$$

implying (7), as claimed. We now have $c(\cdot,\cdot)$ defined on $\mathcal{F}^m_{con} - \mathcal{F}^m_{con}$ and it is easy to show that axioms A1-A4 are satisfied on this class.

The following lemma shows that $c(F, w)$ for $w \in \mathbb{R}^m_{++}$ is uniquely determined by the behavior of F on the box $X^m_{j=1}[0, w_j] \equiv Bw$. In other words, it is enough to restrict ourselves to the box Bw generated by w.

Lemma 2. Suppose that F and G are in $\mathcal{F}^m_{con}$ and let $w \in \mathbb{R}^m_{++}$. If $F(x) = G(x)$ for all $x \in \mathbb{R}^m_+$ s.t. $x \leq w$ then

$$c(F, w) = c(G, w)$$

Proof of Lemma 2. Since $F(x) = G(x)$ for all $0 \leq x \leq w$ then $F(x) - G(x)$ and $G(x) - F(x)$ are both non decreasing for all $0 \leq x \leq w$. By the positivity axiom,

$$c(F, w) \geq c(G, w)$$

and

$$c(G, w) \geq c(F, w)$$

Hence, $c(F, w) = c(G, w)$.■

Definition 4. Denote $\beta = Bw$ the box generated by $w \in \mathbb{R}^m_{++}$. Let $\mathcal{F}^m_{con}(\beta)$ be the set of all continuously differentiable, concave functions F with $F(0) = 0$ restricted to β. Namely, $\mathcal{F}^m_{con}(\beta)$ is generated by $\mathcal{F}^m_{con}$ by simply restricting the domains of every function $F \in \mathcal{F}^m_{con}$ to β.

By Lemma 2, for every $w \in \mathbb{R}^m_{++}$, $c(\cdot,\cdot)$ is defined on $\mathcal{F}^m_{con}(\beta) - \mathcal{F}^m_{con}(\beta)$, where $\beta = Bw$. Next, we define the C^1-norm $\|\cdot\|_1$ on $\mathcal{F}^m_{cd}$ as follows:[6]

$$\|F\|_1 = \sum_{i=1}^{m} \sup \left| \frac{\partial F}{\partial x_i} \right|$$

where the sup is taken over $x \in \beta$. We first show that every monomial of the form

$$P(x_1, ..., x_m) = b x_1^{k_1} x_2^{k_2} ... x_m^{k_m} \tag{8}$$

with $b \geq 0$ and non negative integers k_1,...,k_m (with at least one of them positive) is in $\mathcal{F}^m_{con} - \mathcal{F}^m_{con}$ for every $w \in \mathbb{R}^m_{++}$. The reason is that $P(x_1, ..., x_m)$ is convex and therefore:

$$g(x_1, ..., x_m) = \sum_{i=1}^{m} a_i x_i - P(x_1, ..., x_m)$$

[6] Note that $\|\cdot\|_1$ is indeed a norm, since $\|F\|_1 = 0$ implies that $F = 0$ due to the fact that $F(0) = 0$.

is concave. Since $\sum_{i=1}^{m} a_i x_i$ is also concave it follows that $g \in \mathcal{F}_{con}^m$ and

$$P(x_1, ..., x_m) = \sum_{i=1}^{m} a_i x_i - g(x_1, ..., x_m) \in \mathcal{F}_{con}^m - \mathcal{F}_{con}^m.$$

We next observe that every polynomial is the sum and difference of a finite number of monomials. Since $\mathcal{F}_{con}^m - \mathcal{F}_{con}^m$ is a linear space we also conclude that every polynomial is in $\mathcal{F}_{con}^m - \mathcal{F}_{con}^m$. Next, we note that every function $F \in \mathcal{F}_{cd}^m(\beta)$ can be approximated by polynomials in the C^1−norm (see Hilbert (1953)). Namely, there exists a sequence of polynomials $(P_k(\cdot))_{k=1}^{\infty}$ on $\mathbb{R}^m$ s.t.

$$\|P_k - F\|_1 \to 0, \text{ as } k \to \infty.$$

We will show that $c(P_k, w)$ converges, as $k \to \infty$ and we will define $c(P, w)$ as the limit of $c(P_k, w)$, and the extension of $c(F, w)$ to $\mathcal{F}^m$ (and hence to $\mathcal{F}$) will be completed.

Lemma 3. Let $\overline{w} \in \mathbb{R}_{++}^m$ and let $(P_k)_{k=1}^m$ be a sequence of polynomials on $\mathbb{R}^m$ s.t. $\|P_k\|_1 \to 0$, as $k \to \infty$ on $B\overline{w}$. Then $c(P_k, w) \to 0$, as $k \to \infty$, for every $w \neq 0$ with $0 \leq w \leq \overline{w}$.

Proof of Lemma 3. Since $\|P_k\|_1 \to 0$, as $k \to \infty$, on $B\overline{w}$, by (8) P_k and its partial derivatives converge to zero in the sup norm for every $w \in \mathbb{R}_+^m$ with $w \leq \overline{w}$. This implies that for every $\epsilon > 0$ and for k sufficiently large both $P_k(x_1, ..., x_m) + \epsilon \sum_{i=1}^{m} x_i$ and $\epsilon \sum_{i=1}^{m} x_i - P_k(x_1, ..., x_m)$ are non decreasing functions on $B\overline{w}$. By the positivity and the additivity axioms

$$\begin{aligned} c(P_k, w) + c\left(\epsilon \sum_{i=1}^{m} x_i, w\right) &\geq 0 \\ c\left(\epsilon \sum_{i=1}^{m} x_i, w\right) - c(P_k, w) &\geq 0 \end{aligned}$$

Hence, for k sufficiently large,

$$-c\left(\epsilon \sum_{i=1}^{m} x_i, w\right) \leq c(P_k, w) \leq c\left(\epsilon \sum_{i=1}^{m} x_i, w\right) \tag{9}$$

Note also that for every $F \in \mathcal{F}^m$ we have that $c(\epsilon F, w) = \epsilon c(F, w)$ (using the additivity axiom this is true for any rational number $\epsilon > 0$ and by the positivity axiom it is easy to verify that it is also true for any real number ϵ). Hence, by (9) for k sufficiently large

$$-\epsilon c\left(\sum_{i=1}^{m} x_i, w\right) \leq c(P_k, w) \leq \epsilon c\left(\sum_{i=1}^{m} x_i, w\right)$$

Since $c\left(\sum_{i=1}^{m} x_i, w\right)$ does not depend on k we conclude that $\lim_{k\to\infty} c(P_k, w) = 0$. ■

The following lemma shows that $\lim_{k\to\infty} c(P_k, w)$ exists if $(P_k)_{k=1}^m$ converges to any continuously differentiable function.

Lemma 4. Let $\overline{w} \in \mathbb{R}^m_{++}$ and let $\beta = B\overline{w}$. Suppose that $F \in \mathcal{F}^m_{cd}$ and let $(P_k)_{k=1}^{\infty}$ be a sequence of polynomials on $\mathbb{R}^m$ s.t. $\|P_k - F\|_1 \to 0$, as $k \to \infty$, on β. Then $\lim_{k\to\infty} c(P_k, w)$ exists for every $w \neq 0$ with $0 \leq w \leq \overline{w}$.

Proof of Lemma 4. $(P_k)_{k=1}^{\infty}$ is a Cauchy sequence since it converges (to F). That is, for every $\epsilon > 0$ there exists $N > 0$ s.t. whenever $j, k \geq N$ then $\|P_j - P_k\|_1 < \epsilon$ on β. Let $a_j = c_j\left(\sum_{t=1}^m x_t, w\right)$ and let $a = \sum_{j=1}^m a_j$. Let $\epsilon > 0$ and let $\delta > 0$ be such that $\delta < \min\left(\epsilon, \frac{\epsilon}{a}\right)$. Since $(P_k)_{k=1}^{\infty}$ is a Cauchy sequence there exists $N > 0$ s.t.

$$l, k \geq N \text{ implies } \|P_l - P_k\|_1 < \delta \tag{10}$$

By (10), $P_l - P_k$ and the partial derivatives of $P_l - P_k$ are smaller than δ in the sup norm. Hence, if $l, k \geq N$ both $(P_l - P_k)(x_1, ..., x_m) + \delta\sum_{j=1}^m x_j$ and $\delta\sum_{j=1}^m x_j - (P_l - P_k)(x_1, ..., x_m)$ are non decreasing on β. By the positivity and the additivity axioms (which apply to $\mathcal{F}^m_{con} - \mathcal{F}^m_{con}$) we have that for all $j = 1, ..., m$ and for all $w \neq 0$ with $0 \leq w \leq \overline{w}$,

$$-\epsilon < -\delta c_j\left(\sum_{i=1}^m x_i, w\right) \leq c_j(P_l - P_k, w) \leq \delta c_j\left(\sum_{i=1}^m x_i, w\right) < \epsilon \tag{11}$$

By (11) and the additivity axiom

$$-\epsilon < c_j(P_l, w) - c_j(P_k, w) < \epsilon$$

or

$$|c_j(P_l, w) - c_j(P_k, w)| < \epsilon$$

Consequently, $(c_j(P_k, w))_{k=1}^{\infty}$ is a Cauchy sequence on the real line. Since the real line is a complete space every Cauchy sequence converges to a limit. Hence $(c_j(P_k, w))_{k=1}^{\infty}$ and thus $(c(P_k, w))_{k=1}^{\infty}$ converges to a finite limit. ■

We are now ready to extend $c(\cdot, \cdot)$ to $\mathcal{F}^m_{cd}$. Let $F \in \mathcal{F}^m_{cd}$ and let $w \in \mathbb{R}^m_+$ with $w \neq 0$. Suppose that $\overline{w} \in \mathbb{R}^m_{++}$ s.t. $w \leq \overline{w}$ and let $\beta = B\overline{w}$. There exists a sequence of polynomials $(P_k)_{k=1}^{\infty}$ on $\mathbb{R}^m_+$ s.t. $\|P_k - F\|_1 \to 0$, as $k \to \infty$, on β. By Lemma 4 $(c(P_k, w))_{k=1}^{\infty}$ has a limit. Define

$$c(F, w) = \lim_{k\to\infty} c(P_k, w) \tag{12}$$

We need to show that (12) is well defined and does not depend on the representative sequence $(P_k)_{k=1}^{\infty}$. Suppose that also the sequence $(G_k)_{k=1}^{\infty}$ of polynomials converges to F on $\widehat{\beta} = B\widehat{w}$, where $\widehat{w} \in \mathbb{R}^m_{++}$ and $w \leq \widehat{w}$. Let $\overline{\beta} = \beta \cap \widehat{\beta}$. Then $\|P_k - F\|_1 \to 0$ and $\|G_k - F\|_1 \to 0$ as $k \to \infty$, on $\overline{\beta}$. Thus,

$$\|P_k - G_k\|_1 = \|P_k - F + F - G_k\|_1 \leq \|P_k - F\|_1 + \|F - G_k\|_1 \to 0 \text{ as } k \to \infty, \text{ on } \overline{\beta}$$

and

$$\|P_k - G_k\|_1 \to 0 \text{ as } k \to \infty, \text{ on } \overline{\beta}$$

By Lemma 3, we have that $\lim_{k\to\infty} c(P_k - G_k, w) \to 0$, as $k \to \infty$. By Lemma 4, $\lim_{k\to\infty} c(P_k, w)$ and $\lim_{k\to\infty} c(G_k, w)$ exist and by the additivity of $c(\cdot, w)$ on $\mathcal{F}^m_{con} - \mathcal{F}^m_{con}$

$$\lim_{k\to\infty} c(P_k, w) - \lim_{k\to\infty} c(G_k, w) \to 0 \text{ as } k \to \infty.$$

Consequently,

$$\lim_{k\to\infty} c(P_k, w) = \lim_{k\to\infty} c(G_k, w) \text{ as } k \to \infty,$$

and (12) is well defined.

It is easy to verify that $c(F, w)$, which is now defined for every $F \in \mathcal{F}_{cd}^m$, satisfies the first two axioms of rescaling and consistency. In addition, $c(F, w)$ is linear, namely

$$c(\alpha F + \gamma G, w) = \alpha c(F, w) + \gamma c(G, w)$$

for all real numbers α and γ and all F and G in $\mathcal{F}_{cd}^m$. Furthermore, $c(\cdot, w)$ is weakly positive. Namely, for $w \in \mathbb{R}_{++}^m$ if $F \in \mathcal{F}_{cd}^m$ and F is non decreasing on Bw then $c(F, w) \geq 0$.

We are now ready to complete the proof using the technique in Samet and Tauman (1982). A sketch of the proof is provided in what follows. We next prove the theorem for the one dimensional case (or the single product market).

Lemma 5. There exists a unique non negative measure μ on $([0,1], ß)$ such that for each $w \in \mathbb{R}_{++}$ and for each $F \in \mathcal{F}_{cd}^1([0, w])$

$$c(F, w) = \int_0^1 F'(tw)\, d\mu(t) \tag{13}$$

Proof of Lemma 5. We first prove the lemma for the case with $w = 1$. Let $C[0,1]$ be the set of all continuous real valued functions on $[0,1]$. There is a one to one linear mapping φ from $C[0,1]$ onto $\mathcal{F}_{cd}^1([0,1])$ defined by

$$(\varphi f)(x) = \int_0^x f(t)\, dt \tag{14}$$

for all $f \in C[0,1]$ and all $0 \leq x \leq 1$. Observe that $c(\cdot, 1)$ defines a functional ψ on $C[0,1]$ by[7]

$$\psi(f) = c(\varphi f, 1),\ f \in C[0,1] \tag{15}$$

The additivity and positivity of $c(\cdot, 1)$ on $\mathcal{F}_{cd}^1$ implies the additivity and the positivity of ψ on $C[0,1]$. Note that positivity means that $\psi(f) \geq 0$ whenever $f \geq 0$. This follows from (14) since $f \geq 0$ implies that φf is non decreasing in $\mathcal{F}_{cd}^1([0,1])$. Moreover, additivity of $C[0,1]$ implies that

$$\psi(f+g) = \psi(f) + \psi(g)$$

Using the additivity and positivity of ψ it is easy to verify that $\psi(\alpha f) = \alpha\psi(f)$ for each real number α. Indeed first let $\alpha = \frac{1}{k}$, where k is a positive integer. By additivity we have that

$$\psi(f) = \psi\left(k\frac{1}{k}f\right) = \psi\left(\frac{1}{k}f + ... + \frac{1}{k}f\right) = k\psi\left(\frac{1}{k}f\right)$$

Thus $\alpha\psi(f) = \psi(\alpha f)$. Suppose next that $\alpha = \frac{t}{k}$, where t and k are positive integers. Then by the additivity of ψ, $\psi\left(\frac{t}{k}f\right) = t\psi\left(\frac{1}{k}f\right)$. Since $\psi\left(\frac{1}{k}f\right) = \frac{1}{k}\psi(f)$ we have that

[7] Note that φf is a continuously differentiable function and $(\varphi f)(0) = 0$. Hence $\varphi f \in \mathcal{F}_{cd}^1$ and $c(\varphi f, 1)$ is well defined.

$\psi\left(\frac{t}{k}f\right)=\frac{t}{k}\psi\left(f\right)$. Next suppose that $\alpha=-\frac{t}{k}$. Since $\psi\left(0\right)=0$ we have that $\psi\left(-\frac{t}{k}f+\frac{t}{k}f\right)=\psi\left(0\right)=0$. By the additivity of ψ we also have that $0=\psi\left(-\frac{t}{k}f\right)+\psi\left(\frac{t}{k}f\right)$. Hence,

$$\psi\left(-\frac{t}{k}f\right)=-\frac{t}{k}\psi\left(f\right)$$

and therefore $\psi\left(\alpha f\right)=\alpha\psi\left(f\right)$ for every rational α. Finally, let α be a real number. Let $\left(r_k\right)_{k=1}^{\infty}$ be an increasing sequence of rational numbers and let $\left(s_k\right)_{k=1}^{\infty}$ be a decreasing sequence of rational numbers s.t. $r_k\uparrow\alpha$ and $s_k\downarrow\alpha$, as $k\rightarrow\infty$. Since $\left(\alpha-r_k\right)f\geq 0$ and $\left(s_k-\alpha\right)f\geq 0$ by the positivity of ψ we get that

$$\psi\left(\alpha f-r_kf\right)\geq 0 \text{ and } \psi\left(s_kf-\alpha f\right)\geq 0$$

By the additivity of ψ

$$r_k\psi\left(f\right)=\psi\left(r_kf\right)\leq\psi\left(\alpha f\right)\leq\psi\left(s_kf\right)=s_k\psi\left(f\right)$$

Letting $k\rightarrow\infty$ we have that

$$\alpha\psi\left(f\right)\leq\psi\left(\alpha f\right)\leq\alpha\psi\left(f\right)$$

so that $\alpha\psi\left(f\right)=\psi\left(\alpha f\right)$. Consequently, ψ is linear, namely

$$\psi\left(\alpha f+\gamma g\right)=\alpha\psi\left(f\right)+\gamma\psi\left(g\right)$$

for any real numbers α,γ and any $f,g\in C\left[0,1\right]$. Since ψ is a positive functional on $C\left[0,1\right]$ the Rietz Representation Theorem implies that there exists a unique nonnegative measure μ on $\left(\left[0,1\right],\text{ß}\right)$ s.t.

$$\psi\left(f\right)=\int_0^1 f\left(t\right)d\mu\left(t\right)$$

This together with (14) and (15) imply that

$$c\left(F,1\right)=\int_0^1 F'\left(t\right)d\mu\left(t\right) \tag{16}$$

where

$$F\left(x\right)=\int_0^x f\left(t\right)d\left(t\right)$$

When we vary f over all functions in $C\left[0,1\right]$ we vary F over all $\mathcal{F}_{cd}^{1}\left(\left[0,1\right]\right)$. Therefore, by Lemma 2 (16) holds for every $F\in\mathcal{F}_{cd}^{1}$. Next, let $w>0$ and let $F\in\mathcal{F}_{cd}^{1}$. Define $G\left(x\right)\equiv F\left(wx\right)$ for all $x\in\left[0,1\right]$. By (16),

$$c\left(G,1\right)=\int_0^1 G'\left(t\right)d\mu\left(t\right)$$

By the rescaling axiom, $c(G,1) = wc(F,w)$. Thus, for all $w > 0$

$$c(F,w) = \frac{1}{w}\int_0^1 G'(t)\,d\mu(t) = \int_0^1 F'(tw)\,d\mu(t)$$

and the proof of the lemma is complete.■

Lemma 6. For any polynomial Q on $\mathbb{R}^m$ of the form

$$Q(x_1,...,x_m) = (n_1x_1 + ... + n_mx_m)^l$$

and for any $w \neq 0$

$$c_j(Q,w) = \int_0^1 \frac{\partial Q_j}{\partial x_j}(tw)\,d\mu(t)\,, \text{ for } j = 1,...,m$$

Proof of Lemma 6. Suppose that $n_i > 0$ for all $i = 1,...,m$. Let $L : \mathbb{R} \to \mathbb{R}$ be defined by $L(x) = x^l$. Then

$$Q(x_1,...,x_m) = L\left(\sum_{j=1}^m n_jx_j\right) \tag{17}$$

Let

$$F(x_1,...,x_m) = Q\left(\frac{1}{n_1}x_1,...,\frac{1}{n_m}x_m\right) = L\left(\sum_{j=1}^m x_j\right) \tag{18}$$

By the consistency axiom and by Lemma 5, for every i, $1 \leq i \leq m$,

$$c_j(F,w) = c\left(L, \sum_{k=1}^m w_k\right) = \int_0^1 L'\left(t\sum_{k=1}^m w_k\right)d\mu(t) \tag{19}$$

By (17) and (18)

$$Q(x_1,...,x_m) = F(n_1x_1,...,n_mx_m) \tag{20}$$

By (17), (19) and (20) and by the rescaling axiom,

$$c_j(Q,w) = n_jc_j(F, n*w) = n_j\int_0^1 L'\left(t\sum_{k=1}^m n_kw_k\right)d\mu(t) = \int_0^1 \frac{\partial Q_j}{\partial x_j}(tw)\,d\mu(t)$$

and the proof is complete for all $n_j > 0$ and $j = 1,..,m$. It is easy to extend the proof to the case in which $n_j \geq 0$ for $j = 1,...,m$.■

Lemma 7. For any polynomial P on $\mathbb{R}^m$ and every $w \in \mathbb{R}^m_+$ and $w \neq 0$

$$c_j(P,w) = \int_0^1 \frac{\partial P}{\partial x_j}(tw)\,d\mu(t)$$

Proof of Lemma 7. Any polynomial P on $\mathbb{R}^m$ is of the form

$$P(x_1,...,x_m) = \sum_{k=1}^v \alpha_kQ_k(x_1,...,x_m) \tag{21}$$

where Q_k is of the form given in Lemma 6 (see Aumann and Shapley (1974), p. 41). By the linearity of $c(\cdot, w)$ we have

$$c_j(P, w) = c_j\left(\sum_{k=1}^{v} \alpha_k Q_k, w\right) = \sum_{k=1}^{v} \alpha_k c_j(Q_k, w)$$

By Lemma 6,

$$c_j(P, w) = \sum_{k=1}^{v} \alpha_k \int_0^1 \frac{\partial Q_k}{\partial x_j}(tw)\, d\mu(t)$$

By (21) and by the fact that the integral is linear

$$c_j(P, w) = \int_0^1 \frac{\partial P}{\partial x_j}(tw)\, d\mu(t)$$

as claimed.■

To complete the proof of the theorem, let $F \in \mathcal{F}^m_{cd}$. Let $w \in \mathbb{R}^m_+$ and let $\overline{w} \in \mathbb{R}^m_{++}$ s.t. $\overline{w} \geq w$. Then there exists a sequence of polynomials $(P_k)_{k=1}^{\infty}$ on $\mathbb{R}^m$ s.t. $\|P_k - F\|_1 \to 0$ as $k \to \infty$ on β, where $\beta = B\overline{w}$. By definition and by Lemma 5, for every j, $1 \leq j \leq m$,

$$\begin{aligned} c_j(F, w) &= \lim_{k\to\infty} c_j(P_k, w) = \lim_{k\to\infty} \int_0^1 \frac{\partial P_k}{\partial x_j}(tw)\, d\mu(t) \\ &= \int_0^1 \frac{\partial F}{\partial x_j}(tw)\, d\mu(t) \end{aligned}$$

This completes the proof of the theorem asserting that for every contribution mechanism $c(\cdot, \cdot)$ which satisfies the four axioms there exists a unique non negative measure μ on $([0, 1], ß)$ s.t. for every $w \in \mathbb{R}^m_+$, every $F \in \mathcal{F}^m$ and every i with $1 \leq i \leq m$

$$c_j(F, w) = \int_0^1 \frac{\partial F}{\partial x_j}(tw)\, d\mu(t)$$

Finally, it is easy to verify that each such mechanism satisfies the four axioms. This completes the proof of the theorem. ■

Proof of Proposition 3. Suppose that $x = (x^1, ..., x^n) \in \mathbb{R}^{mn}_+$ is a proper allocation. Then in particular x is Pareto efficient, namely, there exists $(\lambda_i)_{i\in I}$ s.t.:

$$\begin{aligned} x &\in \arg\max_{(y^i)_{i\in I}} \sum_{i=1}^{n} \lambda_i u_i(y^i) \text{ s.t.} \\ \sum_{i\in I} y^i &\leq \sum_{i\in I} w^i = w_a \end{aligned}$$

By the second welfare theorem, x is a Walrasian equilibrium with respect to the case where the initial endowments are x. That is, for every $i \in I$, x^i solves:

$$\begin{aligned} &\max_{y^i \in \mathbb{R}^m_+} u_i(y^i) \text{ s.t.} \\ py^i &= px^i \end{aligned}$$

It is easy to verify (and well known) that up to a positive factor

$$p = \bigtriangledown F\left(w_a\right) = \lambda_i \bigtriangledown u_i\left(x^i\right) \tag{22}$$

Since u_i is homogeneous of degree r for each $i \in I$

$$x^i \bigtriangledown u_i\left(x^i\right) = r u_i\left(x^i\right) \tag{23}$$

Next observe that $F_\lambda\left(w_a\right)$ is also homogeneous of degree r. Since x is a proper allocation

$$\frac{\lambda_i u_i\left(x^i\right)}{\sum_{k\in I}\lambda_k u_k\left(x^k\right)} = \frac{w^i c\left(F, w_a\right)}{\sum_{k=1}^n w^k c\left(F, w_a\right)} \tag{24}$$

where for some nonnegative measure μ on $([0,1], ß)$.

$$c\left(F, w_a\right) = \int_0^1 \nabla F\left(t w_a\right) d\mu\left(t\right) = \frac{1}{r}\nabla F\left(w_a\right)\int_0^1 t^{r-1} d\mu\left(t\right)$$

Let $\gamma = \int_0^1 t^{r-1} d\mu\left(t\right)$. Then by (22)

$$c\left(F, w_a\right) = \frac{\gamma}{r}\nabla F\left(t w_a\right) = \frac{\gamma}{\alpha r} p \tag{25}$$

Consequently, by (24) and (25)

$$\frac{\lambda_i u_i\left(x^i\right)}{\sum_{k\in I}\lambda_k u_k\left(x^k\right)} = \frac{p w^i}{\sum_{k\in I} p w^k} = \frac{p w^i}{p w_a} \tag{26}$$

By (22) and (23)

$$\begin{aligned}\frac{\lambda_i u_i\left(x^i\right)}{\sum_{k\in I}\lambda_k u_k\left(x^k\right)} &= \frac{\lambda_i x^i \nabla u_i\left(x^i\right)}{\sum_{k\in I} x^k \lambda_k \nabla u_k\left(x^k\right)} = \frac{x^i \nabla F_\lambda\left(w_a\right)}{\sum_{k\in I} x^k \nabla F_\lambda\left(w_a\right)} \\ &= \frac{p x^i}{p\sum_{i\in I} x^i} = \frac{p x^i}{p w_a}\end{aligned} \tag{27}$$

By (24), (25), (26) and (27) we have that $px^i = pw^i$ and hence every $i \in I$ maximizes

$$\max_{y^i} u_i\left(y^i\right) \text{ s.t.}$$
$$p y^i \le p w^i$$

This implies that x is a Walrasian equilibrium with respect to w.

References

[1] To be added.