B1. 

a) Derive a Riccati equation that arises in the process of factoring a second order differential operator

\[
\frac{d^2}{dx^2} + P_1(x) \frac{d}{dx} + P_0(x) = \left[ \frac{d}{dx} + a(x) \right] \left[ \frac{d}{dx} + b(x) \right].
\]

b) Solve the corresponding Riccati equation to factor the following differential operator

\[
\frac{d^2}{dx^2} + \frac{x^2 + 1}{x} \frac{d}{dx} + 2.
\]
B2. Obtain a first order perturbative approximation $y(x) = y_0(x) + \varepsilon y_1(x)$ to the initial value problem with a small positive parameter $\varepsilon$

$$y'' + (1 + \varepsilon)y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$ 

Compare the behavior of the perturbative solution at large $x$ with the exact solution. In which $x$-domain is this approximate solution valid?
Consider the matrix $C = (A^T A)^{-1}$, where $A \in \mathbb{R}^{m \times n}$ with rank$(A) = n$.

a) (5 points) Suppose the reduced QR factorization $A = QR$ is available. Show that $C = (R^T R)^{-1}$.

b) (5 points) What is the condition number of $C$ in 2-norm in terms of the singular values of $A$?
Consider $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$.

a) (6 points) Show that
\[
\begin{bmatrix}
AB & 0 \\
B & 0
\end{bmatrix}
\] and
\[
\begin{bmatrix}
0 & 0 \\
B & BA
\end{bmatrix}
\] are similar to each other.

b) (4 points) Use the result in (a) to show that the nonzero eigenvalues of $AB$ are the same as those of $BA$. 