STAT AREA PART I
Mathematical Statistics

NAME: ________________________  ID: ________________________

Instruction: Work all four problems. You may not use notes or any other assistance.
1. Let $S = [0, 1]$, $X \sim \text{U}(0, 1)$, $X(s) = s$, and

$$X_1(s) = s + \text{I}_{[0,1]}(s), \quad X_2(s) = s + \text{I}_{[0,1/2]}(s), \quad X_3(s) = s + \text{I}_{[0,1/3]}(s),
X_4(s) = s + \text{I}_{[0,1/2]}(s), \quad X_5(s) = s + \text{I}_{[1/3,2/3]}(s),
X_6(s) = s + \text{I}_{[1/3,2/3]}(s), \quad \cdots$$

(a) Show that $X_n \xrightarrow{p} X$, but $X_n$ does not converge to $X$ almost surely.

(b) Find a subsequence of the $X_i$’s that converges almost surely to $X$. 
2. Suppose $Y$ follows a multinomial distribution with probabilities

$$\pi = \left( \frac{1}{2} + \frac{\theta}{4}, \frac{1 - \theta}{4}, \frac{1 - \theta}{4}, \frac{\theta}{4} \right).$$

Suppose we observe $y = (n_1, n_2, n_3, n_4) = (125, 18, 20, 34)$. Find the MLE of $\theta$ using EM Algorithm.
3. Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$, where $\sigma^2$ is known. Assume that the prior distribution of $\theta$ is $N(\mu, b^2)$.

(a) Find the Bayes estimator of $\theta$ under the squared error loss.

(b) Conduct a Bayesian test for $H_0: \theta \leq \theta_0$ vs. $H_1: \theta > \theta_0$ based on the posterior distribution obtained in Part (a) using the rejection region

$$\left\{ x : P(\theta \in \Theta_0^c | x) > \frac{1}{2} \right\},$$

where $\Theta_0$ is the null space.
4. Let $X_1, \cdots, X_n$ be iid Bernoulli($p$).

(a) Find the MLE of $p$.

(b) Find the MSE of the MLE obtained in Part (a).

(c) Assume that the prior distribution of $p$ is Beta($1/2, 1/2$) and find the Bayes estimator $\hat{p}$ of $p$ under the squared error loss.

(d) Find the MSE of $\hat{p}$ obtained in Part (c).

(e) Compare the MSE’s obtained in Part (b) and Part (d) for each of $n = 4$ and $n = 400$ and comment on it.