Quantitative Finance Qualifying Exam
2016 Spring

INSTRUCTIONS
You have 4 hours to do this exam.

Reminder: This exam is closed notes and closed books. No electronic devices are permitted. Phones must be turned completely off for the duration of the exam.

PART 1: Do 2 out of problems 1, 2, 3.
PART 2: Do 2 out of problems 4, 5, 6.
PART 3: Do 2 out of problems 7, 8, 9.
PART 4: Do 2 out of problems 10, 11, 12.

All problems are weighted equally.
On this cover page write which eight problems you want graded.
Problems to be graded:

__________________________________________
Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty.
Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.
Name (PRINT CLEARLY), ID number:

__________________________________________
Signature

Stony Brook University
Applied Mathematics and Statistics
1. Time Value of Money

The following deal with computations involving the time value of money.

a) An individual wishes to take out a loan from a bank for $5,000 which will be paid back with a single payment in two years. The bank’s standard interest charge for such a loan is 10% per year compounded continuously. However, the bank also charges a $100 origination and processing fee at the time the loan is made. What is the effective annual interest rate of the loan; i.e., what interest rate is the borrower actually paying for the loan net the fee?

b) What is the effective annual rate for 10% compounded monthly?

c) Frank agrees to pay Marion $10,000 in two years and in return he receives $9,048.37 from her today. As time goes on he realizes that he will be unable to pay her on time and requests that he pay her back $5,000 in two years and the balance of the loan a year later. How much should Marion demand after three years if she is to realize the same rate of interest? Assume continuous compounding in your computations.

d) A bond with a face value of $100,000, a semi-annual coupon of $2,500 and exactly 20 years to maturity has a market yield quoted at 4.0%. Assuming that there is no accrued interest. What is the price of the bond?
2. Portfolio Optimization

Two uncorrelated risk assets have mean returns of 10% and 12%, respectively, and standard deviations of return of 10% and 15%, respectively. The two risk assets are uncorrelated. The risk-free rate of return is 3.5%. Assume that you have one unit of capital to allocate.

a) Assume that there are no restrictions on short selling, what allocation represents the minimum variance portfolio?

b) Assume that there are restrictions on short selling, what allocation represents the maximum expected return portfolio?

c) What allocation of the risk assets represents the tangent portfolio?
3. Binomial Option Pricing

We are given a single-step geometric binomial pricing lattice as the model for the price dynamics of a stock with current price $S(0)$:

Consider a case in which $S(0) = 95$, $\Delta = 0.5$, $u = 1.05$, and the risk-free rate of return $r = 2.5\%$, then...

a) What conditions must be placed on the probabilities \{p_+, p_-\} in order for the model to make sense?

b) Prove that the market represented by the stock and cash is arbitrage free.

c) What is the risk neutral measure $Q = \{q_+, q_-\}$?

d) What is the price of a put option with a strike price $K = 100$ expiring at $\Delta$?
4. Power Law Model

We wish to investigate the lower tail of a return distribution. We normalize the returns $r$ by subtracting them from their median $m_r$ and dividing by their interquartile range $q_r$. Note that this has the effect of transforming returns below the median to positive values:

$$z = \frac{m_r - r}{q_r}$$

Let $Q(z) = \text{Prob}[Z \geq z]$ denote the survival function of $z$. A log-log plot of the survival function for $z \geq 0$ (i.e., the lower part of the return distribution) is shown below.

![Normalized Lower Tail](image)

a) Does the distribution of $z$ display at any point evidence that the lower tail of the distribution follows a power law? Explain what you looked for to determine this.

b) If so, at what point does that behavior emerge? Explain your answer.

c) If there is evidence of a power law in the lower tail, estimate its exponent. Employ a simple approximation but explain how you accomplished it. If not, hypothesize a reasonable return distribution.

d) Based on your work above, define to proportionality the PDF and CDF of the lower tail.

e) What can you say about the existence of the moments of the distribution based on the work above? Explain you answer.
5. Markowitz Portfolio

Assume that returns follow a multivariate Normal distribution with mean vector $\mu$ and positive-definite covariance matrix $\Sigma$. The mean-variance portfolio optimization with unit capital is the quadratic program below. Note that both long and short positions are allowed in this instance.

$$
\mathcal{M} = \min_x \left\{ \frac{1}{2} x^T \Sigma x - \lambda \mu^T x \mid 1^T x = 1 \right\}
$$

where the risk-reward trade-off is controlled by the parameter $0 \leq \lambda$.

Derive a closed form expression for the efficient frontier of the above optimization problem where the optimal portfolio is expressed as a function of $\lambda$. 

6. Copula

a) Let $F_X(x)$ be the CDF of a continuous multivariate random variable $X$ with marginals $F_{X_i}(x_i), i = 1, \ldots, n$ and let Uniform[0, 1] designate the uniform distribution on the unit interval. Show that the random variable $U = F_{X_i}(X_i) \sim \text{Uniform}[0, 1]$.

b) Let $H_Z(z)$ be the CDF of a continuous univariate random variable $Z$. Show that the random variable $H_Z^{-1}(U)$ where $U \sim \text{Uniform}[0, 1]$ realizes a random variable with the same distribution as $Z$.

c) Derive the copula associated with $F_X(x)$, i.e., a function $C(u), u = \{u_1, \ldots, u_n\}$ where $C$ is a multivariate CDF with Uniform[0, 1] marginals, representing the dependence structure of $F_X(X)$ separate from its marginals.

d) Let $G_Y(y_i), i = 1, \ldots, n$ designate the CDFs of continuous univariate random variables $Y_i$. We wish to construct a multivariate distribution $G_Y(y)$ with marginals $G_{Y_i}(y_i)$ and the same dependence structure as $F_X(x)$. Write an expression for $G_Y(y)$ which accomplishes this.
7. Conditional Expectation

Let $Y$ be integrable on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Prove that $\mathbb{E}(Y|\mathcal{G})$ is the best estimate of $Y$ given the information in $\mathcal{G}$. That is, if $X$ is $\mathcal{G}$-measurable, then the variance

$$\mathbb{V}(\text{Err}) \leq \mathbb{V}(Y - X)$$

where $\text{Err} = Y - \mathbb{E}(Y|\mathcal{G})$. 
8. Call Option

For a European call expiring at time $T$ with strike price $K$, the Black-Scholes price at time $t$, if the time-$t$ stock price is $x$, is:

$$c(t, x) = xN(d_+(T - t, x)) - Ke^{-r(T-t)}N(d_-(T - t, x))$$

where

$$d_\pm(\tau, x) = \frac{1}{\sigma \sqrt{\tau}} \left[ \log \frac{x}{K} + \left( r \pm \frac{\sigma^2}{2} \right) \tau \right]$$

(i) Show that the delta is $c_\delta = N(d_+)$

(ii) Show that terminal condition is $\lim_{t \uparrow T} c(t, x) = (x - K)^*$
9. Stochastic Differential Equation

Consider the stochastic differential equation

\[ dX(u) = (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dB(u), \quad u \geq 0 \]  

(SDE1)

where \( a(u), b(u), \gamma(u), \sigma(u), u \geq 0 \) are adapted process with respect to the filtration generated by the Brownian motion \( B(u), u \geq 0 \). Let \( t \geq 0 \) and \( x \in \mathbb{R} \) be fixed. Define

\[ Z(u) = e^{\int_t^u \sigma(v)dB(v) + \int_t^u \left( b(v) - \frac{1}{2} \sigma^2(v) \right) dv}, \quad u \geq t \]

\[ Y(u) = x + \int_t^u a(v) - \sigma(v)\gamma(u)Z(u) dv + \int_t^u \frac{\gamma(u)}{Z(u)} dB(v), \quad u \geq t \]

(i) Show that \( dZ(u) = b(u)Z(u)du + \sigma(u)Z(u)dB(u), u \geq t \).

(ii) Show that \( X(u) = Y(u)Z(u), u \geq t \), solves (SDE1).
10. Merton Model

Consider the Merton model in the time interval \([0, T]\). A firm is financed by one equity with price \(S_t\) and one bond with face value \(B\) maturing at \(T\). The value process of the firm follows a geometric Brownian motion:

\[
dV_t = \mu_V dt + \sigma_V dW_t
\]

where \(\mu_V, \sigma_V\) are given constants and \(W_t\) is a standard Brownian motion. Assume that \(r\) is the risk-free rate and the (deterministic) initial value is \(V_0\).

Compute the loss distribution at horizon \(T\) for a portfolio formed by this stock.
11. Value Process

Consider the Merton model in the time interval \([0, T]\). A firm is financed by one equity and one bond with face value \(B\) maturing at \(T\). The value process of the firm follows a geometric Brownian motion:

\[
dV_t = \mu_v V_t dt + \sigma_v V_t dW_t
\]

where \(\mu_v, \sigma_v\) are given constants and \(W_t\) is a standard Brownian motion. The value process is not observable. Assume we know the initial stock price \(S_0\) and the volatility \(\sigma_s\). Find a system of equations to determine \(V_0, \sigma_v\).
12. Hazard Rate

Consider a probability space \((\Omega, \mathcal{F}, P)\) and \(m\) independent random times \(\tau_i, i = 1, \ldots, n\) defined on this space. Suppose \(\tau_i\) have the same absolutely continuous cdf \(F(t)\).

Determine \(F(t)\) so that the minimum of the \(\tau_i\) has a constant hazard rate function.
Scratch paper 1
Scratch paper 2
Scratch paper 3
Scratch paper 4
Scratch paper 5
Scratch paper 6
Scratch paper 7
Scratch paper 8
Scratch paper 9
Scratch paper 10
Scratch paper 11
Scratch paper 12
Scratch paper 13