Qualifying Exam (June 2016): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.
Do 2 out of problems 1,2,3.
Do 2 out of problems 4,5,6.
Do 3 out of problems 7,8,9,10,11,12,13,14.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

**problems to be graded:**

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Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

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**Signature**
(1). The following LP was solved (using the big M method) and the optimal tableau is given below. $e_1$ and $e_2$ are the excess variables subtracted from the first and second constraints, and $a_i$ is the artificial variable of the $i$th constraint.

$$\begin{align*}
\text{max } \quad & z = 4x_1 + x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \geq 6 \\
& 2x_1 + x_2 \geq 4 \\
& x_1 + x_2 = 3 \\
& x_1, x_2 \geq 0 \\
\end{align*}$$

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(a). Find the dual of this LP and its optimal solution (the objective value and the value of the dual variables). Use the tableau - do not solve from scratch!

(b). Find the range of values of the objective function coefficient for $x_2$ for which the current basis remains optimal.

(c). Find the range of values of $b_1$ for which the current basis remains optimal.

(d). We wish to add to the LP the constraint $x_2 \geq 1.5$, for which the current optimal solution is not feasible. Set up a tableau on which to proceed by the dual Simplex method to find the new optimal solution.

(e). Solve the problem set up in part (d) using the dual simplex method. Note, if you are doing more than 2 pivots, something is wrong!

(2). (a). Suppose that the starting BFS for the LP we are solving is degenerate. Must the next BFS also be degenerate? If so give a short proof. If not, give an example tableau in which the first BFS is degenerate but the second one is not.

(b). Suppose that the current BFS for the LP we are solving is degenerate. Is the value of the objective (z) guaranteed to remain unchanged? If so give a short proof. If not, give an example tableau in which the current BFS is degenerate but the z after the pivot strictly improves.

(c). Suppose we have a BFS that is nondegenerate. Further suppose that an improving nonbasic variable $x_k$ enters the basis. Prove that if the minimum ratio test for choosing a leaving variable has a unique variable achieving the minimum, $x_{B^c}$, then the next BFS is also non-degenerate.

(3). Consider a Transshipment problem $\min \{ \sum_i \sum_j c_{ij} x_{ij} \mid \sum_j x_{ji} - \sum_j x_{ij} = b_i, \; x \geq 0 \}$, where $b_i$ are the supply/demands and $\sum_i b_i = 0$.

(a). Suppose the costs on all arcs are multiplied by a constant $k$ ($c'_{ij} = c_{ij} \cdot k$). Prove that an optimal solution $x^*$ to the original problem is also optimal to the new problem.

(b). Now suppose the costs on all arcs are increased by a constant $k$ ($c'_{ij} = c_{ij} + k$). Does an optimal solution $x^*$ to the original Transshipment Problem remain optimal to the new problem? Prove or give a counterexample.

For the next 2 parts, consider a 2-commodity transshipment problem, in which we have supplies and demands for 2 different products being shipped on the same network, $b^1(i)$ and $b^2(i)$. The variables defined are $x_{ij}^1$, the amount of product 1 shipped from $i$ to $j$, and $x_{ij}^2$ the amount of product 2 shipped from $i$ to $j$. 

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\[ \min \quad z = \sum_{i,j} c_{ij} (x_{ij}^1 + x_{ij}^2) \]
\[ \text{s.t.} \quad \sum_j x_{ij}^1 - \sum_j x_{ji}^1 = b^1(i) \quad \text{for all } i \]
\[ \sum_j x_{ij}^2 - \sum_j x_{ji}^2 = b^2(i) \quad \text{for all } i \]
\[ 0 \leq x_{ij}^1 + x_{ij}^2 \leq u_{ij} \quad \text{for all } (i,j) \]

(c). We showed in class that if all \( b(i) \) and \( u_{ij} \) are integer, and if an optimal solution exists, then an optimal integer solution exists. Is the same true for the 2-commodity transshipment problem? If so, give a short proof. If not, show a counterexample.

(d). We showed in class that the transshipment problem can be solved in strongly polynomial time (meaning time that is polynomial only in \( n \) and \( m \), using the fact that Tardos’ algorithm solves Linear Programming problems in time polynomial in \( n \), \( m \) and the size of the matrix \( A = \text{size}(\sum_{i,j} a_{ij}) \). Does the same method work for the 2-commodity transshipment problem? If so, give a short proof. If not, explain what goes wrong.

(4). Customers arrive at a single server queue according to \( PP(\lambda) \), a Poisson process with rate \( \lambda \). At time 0, there is exactly one customer in the system and he is in service. Service times of all customers are i.i.d. \( \exp(\mu) \). What is the probability that there will be no customer in the system after the second customer has been served?

(5). Consider an \( M/M/1/K \) queuing system, where each customer who enters the system pays \( $a \) as an admission fee and each customer waiting in the system costs \( $c \) per unit time. Compute the long-run net income (revenues − costs) rate of the system as a function of \( K \).

(6). Let \( \{N(t), t \geq 0\} \) be a Poisson process with arrival rate \( \lambda \). Let \( \{C(t), t \geq 0\} \) be the total life process associated with \( \{N(t)\} \), i.e., \( C(t) = S_{N(t)+1} - S_{N(t)} \), where \( S_n \) is the time of the occurrence of the \( n \)th event. For a given \( t > 0 \), compute \( E[C(t)] \).

(7). Consider the randomized incremental method for solving a 2-variable linear program. (Assume the objective function is to find the lowest (min-y) point in the intersection of the \( n \) halfplanes, \( \{h_1, \ldots, h_n\} \).

(a). At certain stages of the algorithm, we are sometimes required to solve a one-dimensional LP. Draw an example of a case in which the addition of the \( i \)th constraint (halfspace \( h_i \)) requires the solution of a one-dimensional LP; give also an example of a case in which the addition of \( h_i \) does not require the solution of a one-dimensional LP. How does the algorithm tell which case occurs? (What test is done on each \( h_i \)?)

(b). Describe briefly what a “one-dimensional LP” problem is, how it is defined in terms of the added constraint \( h_i \), and how it is solved. What is the running time of the solution method? (in big-Oh notation)

(c). What is the probability that, during the solution of a 2-variable LP, we are required to solve a one-dimensional LP upon insertion of halfplane (constraint) \( h_i \) (the \( i \)th constraint inserted during the algorithm)? Why? (give a brief justification)

(d). What is the expected running time of the algorithm (and why)?

(e). Explain how one can determine efficiently if there exists a circle that separates \( n \) “red” points from \( m \) “blue” points in the plane. What is the running time?

(8). Let \( P \) be a simple \( n \)-gon in the plane.

(a). How efficiently (in big-Oh) can one decompose \( P \) into convex polygons using diagonals of \( P \)? (You need not give details of the algorithm.) Here, any decomposition is fine, as long as it uses diagonals and results in convex polygons.

(b). Let \( \chi(P) \) denote the number of convex polygons in a partition of \( P \) (by diagonals) into the fewest possible convex polygons. Give a (generic) example of an \( n \)-gon \( P \) for which \( \chi(P) \geq n - 5 \). (By “generic”,
we mean that it should be clear that your class of examples generalizes to large values of $n$. For your family of examples, what is the value of $\chi(P)$, as a function of $n$?

(c). Describe briefly an algorithm to compute $\chi(P)$ approximately. State what the high-level steps are of the algorithm (you need not give details of the steps), give the algorithm’s efficiency (in big-Oh), and state what the approximation factor is (and sketch why it is what you claim).

(d). Suppose now that the goal is to cover $P$ with a small number of star-shaped polygons. Let $\sigma(P)$ denote the number of star-shaped polygons in a covering of $P$ with the fewest possible star-shaped polygons. How large can $\sigma(P)$ be? (i.e., what can you say about $s(n) = \max_{n \text{-gons}} \sigma(P)$?) How efficiently can you compute a covering of $P$ with at most $s(n)$ star-shaped polygons? Explain briefly.

(9). Consider a Markov decision problem with the state space $X = \{0, 1, \ldots, m\}$, where $m$ is a natural number. At each state $x \in X$, there is a finite set of available actions $A(x)$. If an action $a \in A(x)$ is chosen at a state $x \in X$, then the system moves to a state $y \in X$ with the probability $p(y|x, a)$ and collects the one-step reward $r(x, a)$. Let $p(0|0, a) = 1$ and $r(0, a) = 0$ for all $a \in A(0)$, and $p(0|x, a) > 0$ for all $x = 1, 2, \ldots, m$ and for all $a \in A(x)$. The goal is to find a stationary optimal policy for the expected total reward criterion. Explain how this problem can be solved by linear programming and write the appropriate linear programming formulation. If there are multiple formulations (e.g., primal and dual), you can choose just one of them.

(10). Consider a Markov decision process with a finite state space $X$, compact action sets $A(x)$, transition probabilities $p(y|x, a)$, and one-step rewards $r(x, a)$, where $x, y \in X$ and $a \in A(x)$. Suppose that the functions $p(y|x, a)$ and $r(x, a)$ are continuous in $a \in A(x)$. Is it true that for this Markov decision process there exists a stationary policy maximizing average rewards per unit time? Illustrate your answer with a proof or example.

(11). (a). Let $M_1$ and $M_2$ be two arbitrary matchings in a bipartite graph $G = (N_1 \cup N_2, E)$. Let the nodes of $N_1$ that are matched in $M_1$ be $X (X \subset N_1)$, and the nodes of $N_2$ that are matched in $M_2$ be $Y (Y \subset N_2)$. Show that there exists some matching $M$ in which $X \cup Y$ are all matched.

(b). Let $G = (N, E)$ be a graph, and let $N' \subset N$ be a subset of the nodes that is matched in some matching $M$. Show that there exists a maximum cardinality matching $M^*$ in which each node of $N'$ is matched.

(12). Recall that a strongly connected directed graph is a graph in which a (directed) path exists from every node $i$ to every other node $j$.

(a). Prove that every strongly connected graph on $n$ nodes has a strongly connected subgraph on all $n$ nodes containing at most $(2n - 1)$ arcs.

(b). STRONGLY CONNECTED SUBGRAPH PROBLEM: Given a strongly connected directed graph $G = (N, A)$ and a bound $K$, is there a subset $A' \subset A$ with $|A'| \leq K$ such that $G' = (N, A')$ is strongly connected. Show that this problem is NP-Complete.

(c). Describe an approximation algorithm for the problem in part (b). Your algorithm should run in polynomial time and always produce a feasible answer with $APX(G)$ arcs, such that $APX(G)/OPT(G) \leq 2$ for every strongly connected directed graph $G$. (Make sure to prove that your algorithm is a factor 2 approximation!)

(13). Let $-\infty < a < b < +\infty$ and $\mathcal{M}$ be the set of random variables $X$ satisfying $P(X \in [a, b]) = 1$. Find $C := \sup_{X \in \mathcal{M}} \text{Var}(X)$ and prove your answer.

(14). Let $X = (X_n)_{n=1,2,\ldots}$ be iid random variables with $E\{X_1\} = 0$. Define $Y_0 = 0$ and

$$Y_n = \left(\sum_{i=1}^{n} X_i\right)^2 - nE\{X_1^2\}, \quad n = 1, 2, \ldots$$

Is $Y = (Y_n)_{n=0,1,\ldots}$ a martingale? Prove your answer.