1. Given the first order linear system equations $x' = Ax$, where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

(a). (3 points) Solve it by using eigenvalue method.

(b). (2 points) Find the particular solution with $x^T(0) = (1, 1)^T$.

(c). (2 points) Find $e^{At}$.

(d). (3 points) Find the general solution to the inhomogeneous equation

$$x' = Ax + f, \quad \text{where} \quad f = (e^t, e^{-t})^T$$
2. Consider solving the following equation using power series

\[ x^2 y'' + xy' + \left( x^2 - \frac{1}{4} \right) y = 0. \]

(a). (1 point) Classify the point \( x = 0 \) to the equation.

(b). (4 points) Express the solution as power series, derive the recursive relation of the coefficients for each branch of the solution.

(c). (2.5 points) Find the general solution to the equation.

(d). (2.5 points) Find the relationship between the solution of the given equation and the functions \( \sin x \) and \( \cos x \).
3. Let $Q \in \mathbb{R}^{m \times n}$ be composed of orthonormal column vectors, where $m \geq n$.

(a) (5 points) Show that for any vector $v \in \mathbb{R}^n$, $\|Qv\|_2 = \|v\|_2$.

(b) (5 points) Show that for any vector $u \in \mathbb{R}^m$, $\|Q^T u\|_2 \leq \|u\|_2$. 
4. Consider a singular matrix $A \in \mathbb{R}^{n \times n}$ with rank $n - 1$. Suppose its left and right singular vectors corresponding to the zero singular value are $u_n$ and $v_n$, respectively.

(a) (5 points) Suppose $s \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$, where $s^T v_n \neq 0$ and $t^T u_n \neq 0$. Show that
\[
\begin{bmatrix}
A & t \\
s^T & 0
\end{bmatrix}
\]
is nonsingular.

(b) (5 points) Show that the solution to
\[
\begin{bmatrix}
A & u_n \\
v_n^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
b \\
0
\end{bmatrix}
\]
is a least squares solution to $Ax = b$. In addition $\|x\|_2$ is minimized. (In other words, $x$ is the pseudoinverse solution to the least squares problem $Ax = b$.)