You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.
Do 2 out of problems 1, 2, 3.
Do 2 out of problems 4, 5, 6.
Do 3 out of problems 7, 8, 9, 10.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

problems to be graded:

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Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number**

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**Signature**

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The following LP was solved (using the big M method) and the optimal tableau is given below. $e_1$ and $e_2$ are the excess variables subtracted from the first and second constraints, and $a_i$ is the artificial variable of the $i$th constraint.

$$\begin{align*}
\text{max} & \quad z = 4x_1 + x_2 \\
\text{s.t.} & \quad 3x_1 + x_2 \geq 6 \\
& \quad 2x_1 + x_2 \geq 4 \\
& \quad x_1 + x_2 = 3 \\
& \quad x_1, x_2 \geq 0 \\
\end{align*}$$

(a). Find the dual of this LP and its optimal solution (the objective value and the value of the dual variables). Use the tableau - do not solve from scratch!

(b). Find the range of values of the objective function coefficient for $x_2$ for which the current basis remains optimal.

(c). Find the range of values of $b_1$ for which the current basis remains optimal.

(d). We wish to add to the LP the constraint $x_2 \geq 1.5$, for which the current optimal solution is not feasible. Set up a tableau on which to proceed by the dual Simplex method to find the new optimal solution.

(e). Solve the problem set up in part (d) using the dual simplex method. Note, if you are doing more than 2 pivots, something is wrong!

(2). Consider a Transshipment problem $\min \{ \sum_i \sum_j c_{ij}x_{ij} \mid \sum_j x_{ji} - \sum_j x_{ij} = b_i, \ x \geq 0 \}$, where $b_i$ are the supply/demands and $\sum_i b_i = 0$.

(a). Suppose the costs on all arcs are multiplied by a constant $k$ ($c'_{ij} = c_{ij} \cdot k$). Prove that an optimal solution $x^*$ to the original problem is also optimal to the new problem.

(b). Now suppose the costs on all arcs are increased by a constant $k$ ($c'_{ij} = c_{ij} + k$). Does an optimal solution $x^*$ to the original Transshipment Problem remain optimal to the new problem? Prove or give a counterexample.

(c). Consider the Balanced Transportation Problem $\min \{ \sum_i \sum_j c_{ij}x_{ij} \mid \sum_j x_{ij} = a_i, \sum_i x_{ij} = b_j, \ x \geq 0 \}$ where $a_i$ are the supplies and $b_j$ the demands, $\sum_i a_i = \sum_j b_j$. Suppose the costs on all arcs are increased by a constant $k$ ($c'_{ij} = c_{ij} + k$). Does an optimal solution $x^*$ to the original Balanced Transportation Problem remain optimal to the new problem? Prove or give a counterexample.

(3). Consider the standard form polyhedron $P = \{ x \mid Ax = b, \ x \geq 0 \}$. (As usual, assume that $A$ is an $m$ by $n$ matrix whose rows are linearly independent, and $b$ is a vector of dim $m$.) For each of the following statements, state whether it is true or false. If true, give a short proof; otherwise, provide a counterexample or explanation.

(a) Suppose a given optimal BFS is nondegenerate, and a nonbasic variable $x_j$ has $z_j - c_j = 0$. The optimal solution is not unique.

(b) Again, suppose the given optimal BFS is nondegenerate, and a nonbasic variable $x_j$ has $z_j - c_j = 0$. There must be another optimal BFS.

(c) Suppose the given optimal BFS is degenerate, and a nonbasic variable $x_j$ has $z_j - c_j = 0$. There exists another optimal solution.

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At every optimal solution, no more than $m$ variables can be positive.

Let $x'$ be a feasible solution with exactly $m$ variables that are positive; then, $x'$ is a BFS.

If the LP is degenerate, it may have an infinite number of BFS’s.

Let $Z_1, Z_2, \ldots$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\sigma^2$. Define $T_n = \sum_{i=1}^{n} Z_i$. Let $M$ be a stopping time with respect to the $Z_i$ sequence such that $E[M] < \infty$. Express the variance $Var(T_M)$ in terms of $E[M]$ and $\sigma^2$.

Let $S_n$ be the time of the occurrence of the $n$th event in a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda$. Compute $E[S_{N(t)}]$.

Cars arrive at a parking lot according to a Poisson process with rate 30 cars per hour. The parking times are i.i.d. exponential random variables with mean 2 hours. Suppose the capacity of the parking lot is 100 and all cars finding the lot full are turned away. Compute the proportion of cars that find the lot full upon arrival.

(a). Let $P$ be a simple $n$-gon. Suppose we want to determine if $P$ is 1-guardable with a point guard (i.e., if $g(P) = 1$) and, if so, find a rightmost such guard point. How efficiently can this be done? Explain. Now suppose $P$ is a simple rectilinear polygon with $n$ edges. Describe a very simple method to determine if $P$ is 1-guardable and to compute the locus (region, $R$) of all possible points where a single guard can be placed to see all of $P$. What shape is this region $R$? How efficient is your algorithm?

(b). Let $R = \{p_1, \ldots, p_n\}$ be a set of $n$ red points in the plane; let $B = \{q_1, \ldots, q_m\}$ be a set of $m$ blue points in the plane. Describe how one can determine in time $O(n + m)$ whether or not there exists a line $\ell$ that separates $R$ and $B$ ($R$ on one side of $\ell$, $B$ on the other side). What if the problem is not in 2D, but in 3D?

Let $S$ be a set of $n$ triangles in the plane.

(a). A line $\ell$ that intersects all triangles of $S$ is called a stabber for $S$. Describe an efficient method to decide if a stabber exists for $S$. What is the running time of your method?

(b). How efficiently can you tell if there exists an axis-parallel stabber (vertical or horizontal line) for $S$? (For this problem, you can assume that the triangles are in general position, and you need not worry about various degenerate situations. It suffices to consider lines that “properly stab” the triangles, intersecting them at points interior to the triangles. There is no need to worry about degenerate cases.)

(c). Suppose now that you want to preprocess $S$ to support efficient queries of the following form: For a query point $q$ (in the plane), which triangle of $S$ has the closest vertex to $q$? Describe how you would solve this problem, giving the preprocessing time, storage space, and query time (in big-Oh).

(d). What is the complexity (in big-Oh) of a single face in the arrangement of the $n$ triangles of $S$ in the plane?

Let $F_i(x), i = 1, \ldots, n$ be $n$ cumulative distribution functions. Give an algorithm for generating random variates from the distribution $F(x) = \prod_{i=1}^{n} F_i(x)$ and prove that your algorithm works correctly (assume that the inverse transform method can be applied to easily generate random variates from $F_i$ for each $i$).

The double-exponential distribution has density function

$$f(x) = \frac{1}{2} e^{-|x|} \text{ for all } x \in \mathbb{R}.$$ 

(1) Give the composition algorithm for generating random variates from this distribution.

(2) Derive the inverse transform algorithm for generating random variates from this distribution.
(3) Compare these two algorithms. Which would you prefer?

(11). Consider a directed graph \( G = (V, A) \) \( s, t \in V \), with arc capacities \( u_{ij} \) that are integral or infinite.
(a). Prove that \( v \), the max flow from \( s \) to \( t \), is finite if and only if there is no directed path from \( s \) to \( t \) containing only arcs of infinite capacity.
(b). Now suppose that there are no infinite capacity paths from \( s \) to \( t \). Let \( A^0 \) be the set of arcs with finite capacity, and let \( M = \sum_{(i,j) \in A^0} u_{ij} \). Show that replacing the capacity of each infinite capacity arc by \( M \) does not change the value of the max flow \( v \).

(12). Recall that a strongly connected directed graph is a graph in which a (directed) path exists from every node \( i \) to every other node \( j \).
(a). Prove that every strongly connected graph on \( n \) nodes has a strongly connected subgraph on all \( n \) nodes containing at most \( 2(n - 1) \) arcs.
(b). STRONGLY CONNECTED SUBGRAPH PROBLEM: Given a strongly connected directed graph \( G = (N, A) \) and a bound \( K \), is there a subset \( A' \subset A \) with \( |A'| \leq K \) such that \( G' = (N, A') \) is strongly connected. Show that this problem is NP-Complete.
(c). Describe an approximation algorithm for the problem in part (b). Your algorithm should run in polynomial time and always produce a feasible answer with \( APX(G) \) arcs, such that \( APX(G)/OPT(G) \leq 2 \) for every strongly connected directed graph \( G \). (Make sure to prove that your algorithm is a factor 2 approximation!)

(13). Consider a total-reward negative MDP with countable state set \( X \) and value function \( V \). Let \( V_0(x) := 0 \) for all \( x \in X \), and for \( n = 1, 2, \ldots \) let \( V_n \) denote the \( n \)-th iterate of value iteration obtained from the initial function \( V_0 \). Finally, let \( U_{\infty} := \lim inf_{n \to \infty} V_n \).
(a) Provide an example when \( U_{\infty} \neq V \).
(b) Prove that \( U_{\infty} = V \) if and only if \( U_{\infty} \) satisfies the optimality equation.

(14). Consider a total-reward MDP \( (X, A(\cdot), p, r) \) with finite state and action sets. For a policy \( \pi \) and discount factor \( \alpha \in [0,1] \), let \( v_{\pi}^\alpha(x) \) denote the total expected discounted reward earned under \( \pi \) when the initial state is \( x \). Howard’s policy iteration algorithm terminates with a stationary policy \( \phi \) that satisfies
\[
\phi(x) \in \arg \max_{a \in A(x)} \left\{ r(x, a) + \alpha \sum_{y \in X} p(y|x, a) v_{\phi}^\alpha(y) \right\}, \quad x \in X. \tag{1}
\]
1. Prove that if \( \alpha < 1 \), then \( \phi \) is optimal.
2. Prove that if \( \alpha = 1 \) and the MDP is a positive MDP, then \( \phi \) is optimal.
3. Prove that if \( \alpha = 1 \) and the MDP is a negative MDP, then \( \phi \) may not be optimal. (Hint: Consider the MDP with state set \( X = \{1,2\} \), action sets \( A(1) = A(2) = \{s, g\} \), transition probabilities \( p(1|1, s) = p(2|1, g) = p(2|2, s) = p(1|2, g) = 1, \) and rewards \( r(1, s) = 0, r(1, g) = -1, r(2, s) = 0, \) and \( r(2, g) = -3. \))