APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2016 (January)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A: 1 2 3 4 5
Part B: 6 7 8 9 10

NAME ________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 27th, 2016
Time: 9:00 AM – 1:00 PM
A1. Using the method of residue, calculate the following integral

\[ \int_{\gamma} \frac{dz}{z^6 - 1} \]

for

(a). \( \gamma \) is a circle, \( \gamma : |z - 1| = \frac{1}{2} \) in counter-clock-wise direction.
(b). \( \gamma \) is a circle, \( \gamma : |z - 1| = \frac{3}{2} \) in clock-wise direction.
(c). \( \gamma \) is a circle, \( \gamma : |z| = \frac{1}{2} \) in counter-clock-wise direction.
(d). \( \gamma \) is a circle, \( \gamma : |z| = \frac{3}{2} \) in counter-clock-wise direction.
A2. Evaluate the following integral

\[ I = \int_{0}^{2\pi} \frac{d\theta}{(2 + \sin \theta)^2}. \]
A3.

- Find the maximum of $|f(z)|$ of the following functions on given domains, if exist.
  (a). $f(z) = \frac{z}{2 + z^2}$
    on unit disk $|z| \leq 1$.
  (b). $f(z) = 1 + \sin^2 z$
    on plate $[0, 2\pi] \times [0, 2\pi]$.

- Find the minimum of $|f(z)|$ of the following functions on given domains, if exist.
  (c). $f(z) = az^3 + bz^2 + cz + d$
    on the entire complex plane.
  (d). $f(z) = e^{-z^2}$
    on unit disk $|z| \leq 1$. 
A4. Solve the initial value problem:

\[ u_{tt} - u_{xx} = 0 \]

\[ u_t(x,0) = 0, \quad u(x,0) = \begin{cases} 
\cos \pi(x - 1) & \text{if } 1 < x < 2 \\
0 & \text{otherwise}
\end{cases} \]

Find and draw the solution at \( t = 1, 2, 3 \) respectively for

(a). \( x \in (-\infty, \infty) \).

(b). \( x \in (0, \infty) \) with \( u_x(0, t) = 0 \).

(c). \( x \in (-\infty, 3) \) with \( u(3, t) = 0 \).

(d). \( x \in (0, 3) \) with \( u_x(0, t) = u_x(3, t) = 0 \).
A5. For the following equation and initial condition

\[ u_t + xu_x = 0 \]

\[ u(x, 0) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } 0 \leq x < 4 \\
0 & \text{if } x \geq 4 
\end{cases} \]

(a). Draw characteristics on the \( x-t \) plane.

(b). Draw solution on the \( u-x \) plane at \( t = 1 \) and \( t = 2 \).

(c). Change equation to

\[ u_t + 4u^3u_x = 0 \]

using the Rankine-Hugoniot condition to find shock speed.

(d). Draw solution of (c) at \( t = 1 \) and \( t = 2 \).
B6. Consider the problem

\[ \min f(x, y) = \left( x - 1 \right)^2 + \left( y - 2 \right)^2 \text{ subject to } \left( x - 1 \right)^2 = 5y. \]

a) (3 points) Compute the Lagrange function \( \mathcal{L}(x, y, \lambda) \) associated with this constrained minimization problem.

b) (4 points) Describe the algorithm of Newton’s method for solving the nonlinear equation \( \nabla \mathcal{L}(x, y, \lambda) = 0 \) to obtain the critical points \( (x_*, y_*, \lambda_*) \). Show one step of the Newton’s method the initial guess \( (x_0 = 1, y_0 = 2, \lambda_0 = 0) \).

c) (3 points) Is \( (x_*, y_*, \lambda_*) \) a minimum, maximum, or saddle point of \( \mathcal{L} \)? Why?
B7. Consider the initial value problem

\[ y' = f(t, y), \quad y(0) = y_0, \]

and a linear multistep method of the form

\[ y_{n+1} = \alpha y_n + \beta y_{n-1} + h \gamma f(t_{n-1}, y_{n-1}), \]

where \( h \) is the time step.

a) (6 points) Choose the constants \( \alpha, \beta, \) and \( \gamma \), so that the order of the method is as high as possible.

b) (4 points) Derive the condition under which the resulting method is convergent.
Consider the BTCS scheme for a scalar linear advection equation on infinite interval. Show that the following property holds for this scheme

$$\|u^{n+1}\|_{\Delta,2} \leq \|u^n\|_{\Delta,2}$$

for all $n > 0$. 
B9.

a) Formulate the Law-Wendroff theorem on conservative methods for nonlinear hyperbolic PDE’s.

b) What are limitations of this theorem and what is the role of TV-stability (nonlinear stability) in resolving them? Give definition of the TV-stability.
B10. Consider an advection-diffusion equation

\[ u_t + au_x = \nu u_{xx}, \]

Using the discrete Fourier transform, perform stability analysis of the following scheme

\[ u_n^{n+1} + \frac{a \Delta t}{\Delta x} (u_{k+1}^{n+1} - u_k^{n+1}) - \frac{\nu \Delta t}{\Delta x^2} (u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}) = u_n^n. \]

Consider cases of $a < 0$ and $a > 0$. 