Common Qualifying Exam Part B
Computational and Applied Math
May 2015

Solve three out of four problems

1. For the following Sturm-Liouville problem

\[ y'' + \lambda y = 0 \]

find the eigen values and corresponding eigen functions satisfying the following boundary conditions (in case you cannot solve for the eigen value, write down the equation for which the eigen values must satisfy).

(a) (2.5 points) \( y(0) = 0 \), \( y(L) = 0 \),
(b) (2.5 points) \( y'(0) = 0 \), \( y'(L) = 0 \),
(c) (2.5 points) \( y(0) = 0 \), \( y'(L) = 0 \),
(d) (2.5 points) \( y(0) + y'(0) = 0 \), \( y(L) - y'(L) = 0 \),
2. The following system of equations describes dynamical evolution of two species in an ecological model

\[ \begin{align*}
    x' &= 30x - 2x^2 - 3xy \\
    y' &= 20x - 4x^2 - 3xy
\end{align*} \]

where the prime is the derivative with respect to time.

(a) (2.5 points) Find all the critical points of the system.

(b) (2.5 points) Find the stability of each critical point.

(c) (2.5 points) Draw the phase plane portrait.

(d) (2.5 points) Classify the system as co-existence or single-survival.
3. Let \( A \in \mathbb{R}^{m \times n} \) (\( m > n \)), \( w \in \mathbb{R}^m \), and define

\[
B = [A \mid w].
\]

(a) (3 points) Show that the largest singular value of \( B \) is greater than or equal to that of \( A \), i.e., \( \sigma_1(B) \geq \sigma_1(A) \).

(b) (3 points) Show that the smallest singular value of \( B \) is smaller than or equal to that of \( A \), i.e., \( \sigma_{\text{min}}(B) \leq \sigma_{\text{min}}(A) \).

(c) (4 points) Let \( A = QR \) and \( B = \tilde{Q}\tilde{R} \) be the reduced QR factorizations of \( A \) and \( B \), respectively. Show that the condition number of \( R \) in 2-norm is smaller than or equal to that of \( \tilde{R} \), i.e., \( \kappa_2(R) \leq \kappa_2(\tilde{R}) \).
4. Suppose $H \in \mathbb{R}^{n \times n}$ is upper Hessenberg.

(a) (3 points) Show that if $T \in \mathbb{R}^{n \times n}$ is upper triangular, then $HT$ and $TH$ are also upper Hessenberg.

(b) (3 points) Show that $HT$ and $TH$ have the same eigenvalues. You may assume either $H$ or $T$ is nonsingular for simplicity.

(c) (4 points) Suppose $H$ is nonsingular, and let $PH = LU$ be the Gaussian elimination with partial pivoting of $H$. Show that $H_1 = UPTL$ is upper Hessenberg and is similar to $H$. 